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References to reviews in Mathematical Reviews before volume 20 (1959) are by volume and page number, as MR **19**, 532: from volume 20 on, by volume and review number, as MR **20** #4387. Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

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Mathematical Reviews

Vol. 22, No. 7A July, 1961 Reviews 5533–5994

GENERAL

5533:

Newman, James R. (Editor). ★The world of mathematics: A small library of the literature of mathematics from A'h-mosé the Scribe to Albert Einstein, presented with commentaries and notes. 4 vols. Simon and Schuster, New York, 1956. xviii + vii + vii + vii + 2535 pp. $9.95.

Paperbound; otherwise identical with the edition reviewed in MR 18, 453.

5534:

★American Mathematical Society Translations. Series 2, Vol. 15. American Mathematical Society, Providence, R.I., 1960. iii + 349 pp. $5.00.

Nine articles, translated from the Russian, on foundations [#5572], algebra [#5651, 5682], topological groups [#5690], complex variable theory [#5733] and topology [#5960, 5966, 5967, 5982].

5535:

★English-Russian Russian-English electronics dictionary. Department of the Army Technical Manual TM 30-545. U.S. Government Printing Office, Washington, D.C., 1956. 944 pp. $3.50.

22,000 Russian terms in electronics and telecommunication.

5536:

★II Всесоюзная топологическая конференция. [Second all-union topological conference]. 5–10 October, 1959. Abstracts of reports. Izdat. Akad. Nauk GSSR, Tbilisi, 1959. 53 pp.

5537:

★Proceedings of the International Congress of Mathematicians, 14-21 August, 1958. Edited by J. A. Todd. Cambridge University Press, New York, 1960. lxiv + 573 pp. $12.50.

This is the official record of the Edinburgh congress and contains the complete texts of 17 one-hour and 33 half-hour addresses (which will be reviewed individually). See also #5557, #5558 below.

5538:

Nachtergaele, Jean M. Tendances actuelles de l'enseignement des mathématiques en humanités. Rev. Questions Sci. (5) 21 (1960), 485–492.

5539:

Aczél, J. A look at mathematical competitions in Hungary. Amer. Math. Monthly 67 (1960), 435–437.

A brief description with illustrations of the Kürschák, the Dániel Arany, and the Schweitzer Memorial competition. The latter is for university students.

R. C. Buck (Madison, Wis.)

5540:

Smirnow, W. I. ★Lehrgang der höheren Mathematik. Teil II. Dritte, verbesserte Aufl. Hochschulbücher für Mathematik, Bd. 2. VEB Deutscher Verlag der Wissenschaften, Berlin, 1960. xii + 583 pp. DM 29.50.

Differential equations, general integral calculus, vector analysis, differential geometry, Fourier series; see MR 17, 716 for 1st German edition. This edition has some typographical corrections, and author's changes in the chapters on differential equations. For other volumes, Russian or German, of this series, see MR 6, 42; 9, 574; 14, 145; 16, 663; 17, 716, 833; 20, 979.

5541:

Steinhaus, H. ★Mathematical snapshots. New edition, revised and enlarged. Oxford University Press, New York, 1960. viii + 328 pp. $6.75.

Some 25% larger than the last American edition [1950]. The author asserts in the preface that his intention is to prove that (1) mathematics is concerned with the real world and (2) mathematics is universal.

5542:

Fadiman, Clifton (Editor). ★Fantasia mathematica: Being a set of stories, together with a group of oddments and diversions, all drawn from the universe of mathematics. Simon and Schuster, New York, 1958. xix + 299 pp. $4.95.

An anthology of serious literature, science fiction and whimsy.

HISTORY AND BIOGRAPHY

See also 5533.

5543:

Hofmann, J. E. Über eine Euklid-Bearbeitung, die dem Albertus Magnus zugeschrieben wird. Proc. Internat. Congress Math. 1958, pp. 554–566. Cambridge Univ. Press, New York, 1960.

Cod. 80/45, fol. 105^{v}–145^{r} in the Dominikaner-Bibliothek

Vienna contains a treatment of the books I to IV of Euclid's Elements in Latin, attributed to some Albertus who is very probably identical with Albertus Magnus. The manuscript (MS) has been studied and prepared for edition by the author's wife, Mrs. Josepha Hofmann, and the author here reports on some of its important aspects for the history of mathematics. Written presumably between 1262 and 1265, it is even possible that the MS represents an autograph of Albertus Magnus himself. Some strong arguments to support this opinion are presented.

It is clear from the text that Albertus was familiar with the Latin translations made by Boetius and Adelhard von Bath from Arabic sources. There is no hint to the famous translation of Campanus—a reason to believe that it was written later than that of Albertus. The most important of the numerous Arabic sources referred to in the manuscript are Alfârâbî and Annairîzî, and, of course, many Greek authors are quoted.

The MS opens with a philosophical introduction, gives definitions of the branches of mathematics, explanations of technical terms (important to determine the sources!), before it proceeds to the subject. The author mentions some interesting details, before closing his well-documented paper with a brief discussion of the Jordanus question and the Campanus edition.

C. J. Scriba (Toronto)

5544:

Grant, Edward. Nicole Oresme and his De proportionibus proportionum. Isis **51** (1960), 293–314.

The function introduced by Thomas Bradwardine in his *Tractatus de proportionibus* (1328) to relate forces, resistances and velocities gained widespread acceptance in the two subsequent centuries. It gave rise i.a. to varied mathematical-physical discussions which came to be designated under the heading of proportio proportionum. The author contends that the work of Nicole Oresme, *De proportionibus proportionum* (about 1360), contains applications and extensions of Bradwardine's ideas in directions unthought of by the latter and that these extensions were an outcome of an attempt to furnish a special mathematical foundation for treating a certain kind of proportion utilized approximately by Bradwardine thirty years earlier. The subject of proportions of proportions proved also to be applicable to celestial motions. Oresme concluded from his discussion that it was probable that any two proposed motions of celestial bodies were incommensurable and most probable that some motion of the heavens was incommensurable to some motion of another sphere. This conclusion served him as an important weapon in the fierce combat he had to wage against the influence of the court astrologers on his royal friend Charles V.

E. J. Dijksterhuis (Bilthoven)

5545:

Busulini, Bruno. Le figure analoghe di Bartolomeo Sovero. Accad. Patavina Sci. Lett. Arti. Atti Mem. **70** (1957/58), no. 2, 35–88.

The author reports on Book V of the work *Curvi ac recti proportio* (1630) by Bartolomeo Sovero (1577–1629), who from 1623 until his death occupied the chair of Galileo in the university of Padua. He proves to be a precursor of the geometry of indivisibles and of the method of generation of the various conic sections from a circle. These curves are obtained by a certain transformation which is called "proportional parallel movement". It is elucidated by an algebraic formulation. The work of Sovero gave rise to two important polemics, one with the first successor of Galileo in Padua, Gloriosi; the second between Guldin and Cavalieri on the subject of the latter's originality.

E. J. Dijksterhuis (Bilthoven)

5546:

Boyer, Carl B. Mathematicians of the French Revolution. Scripta Math. **25** (1960), 11–31.

5547:

Rychlík, Karel. Berechnung der Grundzahl *e* der natürlichen Logarithmen. Časopis Pěst. Mat. **85** (1960), 37–43. (Czech. Russian and German summaries)

In 1890, Bohumír Tichánek (1868–1956) of Czechoslovakia computed *e* to 225 decimals with the help of continued fractions, by a method developed by his teacher F. J. Studnička. His result, according to the author, was at that time the most accurate in existence.

S. R. Struik (Cambridge, Mass.)

5548a:

Hua, Lo-ken [Hua, Loo-Keng]. A brief review of mathematical investigations in China for the last decade. Uspehi Mat. Nauk **15** (1960), no. 3 (93), 193–201. (Russian)

5548b:

Mathematics Group. Research works in mathematics in China from 1949 to 1959. Sci. Sinica **8** (1959), 1218–1228.

The "Mathematics Group" are L. K. Hua, Kwan Chao-chih, Tien Fang-tseng, Tuan Hsio-fu and Cheng Min-teh. These two articles are essentially the same as that by L. K. Hua in Kexue Tongbao (Scientia) **18** (1959), 565–567, translated in Amer. Math. Soc. Not. **6** (1959), 724–730.

5549:

Look, K. H. A study of the theory of functions of several complex variables in China during the last decade. Sci. Sinica **8** (1959), 1229–1237.

This article was reprinted in Amer. Math. Soc. Not. **7** (1960), 155–163.

5550:

Su, Buchin; Ku, Chao-hao. The developments of differential geometry in China for the past ten years. Sci. Sinica **8** (1959), 1238–1242.

This article was reprinted in Amer. Math. Soc. Not. **7** (1960), 163–168.

5551:

Mach, Ernst. ★The science of mechanics: A critical and historical account of its development. Translated by Thomas J. McCormack. New introduction by Karl

Menger. 6th ed., with revisions through the 9th German ed. The Open Court Publishing Co., La Salle, Ill., 1960. xxxii + 634 pp. $6.00.

The first German edition of *Die Mechanik in ihrer Entwicklung historisch-kritisch dargestellt* appeared in 1883, and the last in Mach's lifetime in 1912; the present publisher brought out the first American edition in 1893. The introduction by Menger (himself a member of the initially Mach-oriented Vienna Circle) is a critique of Mach's philosophy, influence and limitations; he records in particular his belief that a basic need in present-day physics is a theory of geometrical microstructure. [Cf. #5552 and #5955, 5956 below.]

5552:

Palter, Robert M. ★Whitehead's philosophy of science. University of Chicago Press, Chicago, Ill., 1960. xv + 248 pp. $7.50.

The author gives a concise introduction to Whitehead's philosophy of science with special reference to his treatment of the constants of externality and the method of extensive abstraction and the theory of objects. For the pure mathematician, the most interesting of these topics is the method of extensive abstraction. The author indicates the relation of Whitehead's work to Menger's "Topology without points" [Rice Inst. Pamphlet 27 (1940), 80–107; MR 3, 135], and the reviewer was struck with the fact that Whitehead's method of extensive abstraction is an anticipation of Cartan's theory of filters. *G. Temple* (Oxford)

5553:

Huxley, G. L. The mathematical work of Edmond Halley. Scripta Math. 24 (1959), 265–273.

5554:

Eisele, Carolyn. Charles S. Peirce nineteenth century man of science. Scripta Math. 24 (1959), 304–324.

5555:

Huxley, G. L. The geometrical work of Christopher Wren. Scripta Math. 25 (1960), 201–208.

5556:

Hemelrijk, J. The statistical work of David van Dantzig (1900–1959). Ann. Math. Statist. 31 (1960), 269–275.

5557:

Davenport, H. The work of K. F. Roth. Proc. Internat. Congress Math. 1958, pp. lvii–lx. Cambridge Univ. Press, New York, 1960.

Address on the occasion of the Fields medal presentations.

5558:

Hopf, H. The work of R. Thom. Proc. Internat. Congress Math. 1958, pp. lx–lxiv. (Text in German) Cambridge Univ. Press, New York, 1960.

Address on the occasion of the Fields medal presentations.

5559:

Aleksandrov, P. S.; Mişcenko, E. F. Lev Semenovič Pontryagin: on his 50th birthday. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) 14 (1960), no. 1 (32), 203–207. (Romanian)

Translation of a Russian original [Uspehi Mat. Nauk 14 (1959), no. 3 (87), 195–202; MR 22 #3669].

5560:

Vişik, M. I.; Liusternik, L. A. Sergeĭ L'vovič Sobolev: on his 50th birthday. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) 14 (1960), no. 1 (32), 208–215. (Romanian)

Translation of a Russian original [Uspehi Mat. Nauk 14 (1959), no. 3 (87), 203–214; MR 22 #3668].

5561:

Szegö, Gabor. Leopold Fejér: in memoriam. Bull. Amer. Math. Soc. 66 (1960), 346–352.

5562:

Khayyam, Omar. Discussion of difficulties in Euclid. Translated by Ali R. Amir-Móez. Scripta Math. 24 (1959), 275–303.

A "proof" of the parallel postulate and discussion of ratios and irrationality. The translation is from a book which was claimed to be copied directly from Khayyam's handwriting; the translator has appended some footnotes on obscurities.

5563:

Tonelli, Leonida. ★Opere scelte. Vol. I: Funzioni di variabile reale. A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1960. vi + 604 pp. (1 plate) L. 6000.

This is the first volume of a projected four-volume collection of the author's works. It contains 46 papers on the theory of functions of a real variable, and a biographical and bibliographical introduction by S. Cinquini.

5564:

Peano, Giuseppe. ★Formulario mathematico. Riproduzione in fac-simile dell'edizione originale. Con introduzione e note di Ugo Cassina e col contributo del comune di Cuneo. Edizioni Cremonese, Rome, 1960. xlviii + xxxvi + 465 pp. (16 plates) L. 5000.

This is a facsimile reproduction of the "*Formulario Mathematico*, edito per G. Peano, editio V (Tomo V de Formulario completo), Torino, fratres Bocca editores, 1908", which G. Peano wrote and personally composed between 1903 and 1908. The original work is written in Peano's own language, latino sine flexione, easily understood by any educated West-European, and consists of:

Preface (12 pp.); List of symbols, alphabetical index, references, corrections (20 pp.); Ch. 1, Mathematical logic (22 pp.); Ch. 2, Arithmetic (44 pp.); Ch. 3, Algebra (90 pp.); Ch. 4, Geometry (44 pp.); Ch. 5, Limits (62 pp.); Ch. 6, Differential calculus (62 pp.); Ch. 7, Integral calculus (48 pp.); Ch. 8, Applications to geometry and complements (71 pp.); Indices and figures (22 pp.).

The volume is completed by a Preface and an Introduction by U. Cassina, and by 14 pages of notes and corrections due to various authors, including Peano.

LOGIC AND FOUNDATIONS

See also 5564, 5673.

5565:

Новиков, П. С. [Novikov, P. S.] ★Элементы математической логики. [Elements of mathematical logic.] Matematičeskaya Logika i Osnovaniya Matematiki. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959. 400 pp. 11.05 r.

The first four chapters of this treatise contain a treatment along traditional lines of elementary logic, including Gödel's completeness theorem. In Chapter 5 a formal system of arithmetic with primitive recursive functions is introduced. In the last chapter the consistency of this system without complete induction is proved by a reduction method, which also yields a proof for the independence of the axiom of complete induction.

A. Heyting (Amsterdam)

5566:

Moh, Shaw-kwei. *N*-generalizable, intuitionistic, codenial, pseudo-modal and co-Δ systems. Acta Math. Sinica 9 (1959), 389–412. (Chinese. English summary)

A system (of propositional calculus) is N-generalizable [N-introducible] if the result of replacing each part NX by CXp [CXp by NX] in each theorem is again a theorem. Heyting's system is regarded as unsatisfactory because it is not N-generalizable on account of the theorem $CpCNpq$, which is not in the author's recommended systems I4 and I3. I3 excludes also $CpCqp$. A system S is intuitionistic if: $\vdash_S NNX \leftrightarrow \vdash_M X$ and $\vdash_S NX \leftrightarrow \vdash_M NX$, M being the classical system. It is pseudo-modal if: $(\vdash_S X$ or $\vdash_S \Delta X) \leftrightarrow \vdash_M X$, ΔX being short for $CNXX$. Two systems S and T are co-denial if: $(Ei)\vdash_S N^i X \leftrightarrow (Ej)\vdash_T N^j X$, i and j of same parity; co-Δ if: $(Em)\vdash_S \Delta^m X \leftrightarrow (Ek)\vdash_T \Delta^k X$. Using these concepts, the author establishes many properties of various systems and proves, e.g., "Among the intuitionistic systems, the co-denial systems, the pseudo-modal systems or the co-Δ systems, there exists at most one which is N-generalizable and N-introducible."

Hao Wang (Oxford)

5567:

Prawitz, Dag. An improved proof procedure. Theoria (Lund) 26 (1960), 102–139.

According to Herbrand's theorem, if A is a formula of the predicate calculus in prenex normal form, then the sequent $\rightarrow A$ is valid if and only if it can be proved from a tautological formula of the propositional calculus using three specified rules PR 1–3 for introducing quantifiers. Let TR 1–3 be the converses of these rules and let $\rightarrow A_1, A_2, \cdots, A_n$ be a sequent of quantifier-free formulas derived from A by applications of TR 1–3. It can be shown that if A is in prenex normal form, then it is valid if and only if there is a valid such sequent of associated formulas. The author here describes a method for finding such a sequent if one exists which includes criteria for determining the most reasonable point at which to apply one of the rules TR 1–3. The general procedure is defined for sequents of the form $\Gamma \rightarrow \theta$ where Γ and θ each includes a list of formulas. The procedure includes preparatory steps to transform the sequent into a form for which a cycle of applications of the TR rule is defined. Branch points within the main cycle lead to subcycles, criteria for which are provided. The procedure is meant to be realizable in an electronic computer. The second part of the paper begins with section 6, which is devoted to discussions of the soundness and completeness of the procedure. Section 7 concerns modifications of the proof method in special cases, and section 8 deals with formulas that are not provable.

E. J. Cogan (Bronxville, N.Y.)

5568:

Farah, Edison. Some propositions equivalent to the axiom of choice. Bol. Soc. Mat. São Paulo 10 (1955), 1–65 (1958). (Portuguese)

After laying his ground out firmly by describing in the early sections the fundamental notions of Zermelo-Fraenkel set theory, including properties of relations, functions, and set operations, and after providing an introduction to the notion of a topology on a set, to filters and ultra-filters, and to compact spaces, the author considers seven statements: (1) the axiom of choice; (2) the generalized distributive law of intersection over union; (3) Zorn's lemma; (4) the well-ordered property of Zermelo; (5) the order trichotomy among cardinal numbers; (6) the equivalence of infinite sets with their Cartesian squares; and (7) the property that every filter on a set E is contained in an ultra-filter on E. Some modifications in the classical proofs of the equivalence of (1) to (3)–(6) are presented. The author establishes the equivalence of (1) to both (2) and (7). The paper concludes with remarks on the relation of the axiom of ultra-filters to Tychonoff's theorem [J. L. Kelley, Fund. Math. 37 (1950), 75–76; MR 12, 626].

E. J. Cogan (Bronxville, N.Y.)

5569:

Kleene, S. C. Mathematical logic: constructive and non-constructive operations. Proc. Internat. Congress Math. 1958, pp. 137–153. Cambridge Univ. Press, New York, 1960.

In this lecture the main negative results of the theory of recursive functions and of its application to logic, including the undecidability results of Church and Gödel, and the relations between them are discussed. Proofs are given when they can be briefly stated. The lecture ends with a discussion of the arithmetical and hyperarithmetical hierarchies.

A. Heyting (Amsterdam)

5570:

Mučnik, A. A. Solution of Post's reduction problem and of certain other problems in the theory of algorithms. Trudy Moskov. Mat. Obšč. 7 (1958), 391–405. (Russian)

As is well known, Post's problem [Bull. Amer. Math. Soc. **50** (1944), 284–316; MR **6**, 29] was solved independently by Friedberg [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 236–238; MR **18**, 867] and Mučnik [Dokl. Akad. Nauk SSSR **108** (1956), 194–197; MR **18**, 457] in 1956. Both authors published their proofs in outline only. In the paper under review, the author gives for the first time a detailed account of his construction of two recursively enumerable (r.e.) sets neither recursive in the other.

The first eight pages of the paper are devoted to preliminary motivational material concerning the history of unsolvable decision-problems and the work of the Soviet school of recursion-theorists (Medvedev, Mučnik himself, Novikov, Trahtenbrot and Uspenskiĭ) on matrix representation of partial recursive operators, universal partial recursive operators and mass-problems. It contains sufficient material (not all of it directly related to the author's main result) to make the paper virtually self-contained. The lucidity and wide effortless coverage of this expository part, together with the interest of the main result, give the paper high priority for anyone contemplating an anthology of translations of Soviet papers on logic.

The second section, in which the incomparable r.e. sets are constructed in detail, is by no means such easy reading. The reviewer has the impression that the proof is nearer to Friedberg's proof than to the author's earlier one; however, a detailed comparative investigation of these three proofs would be necessary to substantiate this impression.

The idea of all the proofs, though highly original, is basically simple—one feels that there is one radically new idea plus a great deal of notational complication. This statement is in no way meant to be derogatory—it applies to Friedberg's proof as well as to Mučnik's two proofs, and the isolation of the new idea is tantalizing indeed. It is presumably some kind of effective category argument. The mere complexity of the notation will probably prevent any substantial progress in this area till this method is isolated.

The author states at the end that he has also proved the existence of an r.e. sequence of r.e. sets which are independent in the sense of Kleene and Post [Ann. of Math. (2) **59** (1954), 379–407; MR **15**, 772]. The reviewer has confirmed this in detail; the amount of routine labor involved supports the impression mentioned in the preceding paragraph. *J. Myhill* (Stanford, Calif.)

5571:

Rogers, Hartley, Jr. Computing degrees of unsolvability. Math. Ann. **138** (1959), 125–140.

Let N be the set of all non-negative integers; sets (i.e., subcollections of N) are denoted by capital Latin letters; $\emptyset$ stands for the empty set, and $\bar{A}$ for the complement of A. The relation "A is 1-1 reducible [Turing reducible] to B" is written $A \leq_1 B$ [resp. $A \leq_T B$]. Mutual 1-1 reducibility [T-reducibility] is denoted by $\equiv_1$ [resp. $\equiv_T$]. The sets A and B are isomorphic ($A \equiv B$) if there is a recursive permutation of N which maps A onto B. According to J. Myhill [Z. Math. Logik Grundlagen Math. **1** (1955), 97–108; MR **17**, 118] $A \equiv_1 B$ if and only if $A \equiv B$. Let $\phi_0(n), \phi_1(n), \cdots$ be any numbering of the partial recursive functions of one variable which is fully effective in the sense of the author's paper J. Symb. Logic **23** (1958), 331–341 [MR **21** #2585]. A similar enumeration $\phi_0{}^A(n)$, $\phi_1{}^A(n), \cdots$ is discussed for all partial functions of one variable which are recursive in A. The ranges of ϕ_x and $\phi_x{}^A$ are denoted by W_x and $W_x{}^A$ respectively. Put $A' = \{x \mid x \in W_x{}^A\}$; this jump operation has the essential property: $A \leq_T B$ if and only if $A' \leq_1 B'$; thus $A \equiv_T B$ if and only if $A' \equiv B'$. Let $S^{(0)} = \emptyset$, $S^{(n+1)} = (S^{(n)})'$, $O^{(n)} =$ the T-degree of $S^{(n)}$; then $\{O^{(n)}\}$ is a strictly increasing sequence of T-degrees. For the notions of a recursively enumerable (r.e.), recursive, simple, hypersimple, creative set, see E. L. Post [Bull. Amer. Math. Soc. **50** (1944), 284–316; MR **6**, 29] and Myhill [loc. cit.]. A complete set is an r.e. set which is T-equivalent to a creative set.

This paper studies the following six sets: $A_1 = \{x \mid W_x$ is recursive$\}$; $A_2 = \{x \mid W_x$ is simple$\}$; $A_3 = \{x \mid W_x$ is hypersimple$\}$; $A_4 = \{x \mid W_x$ is creative$\}$; $A_5 = \{x \mid W_x$ is complete$\}$; $A_6 = \{x \mid W_x$ contains infinitely many y for which W_y is infinite$\}$. Main results: $A_1 \equiv A_4 \equiv S^{(3)}$; $A_2 \equiv A_3 \equiv \overline{S^{(3)}}$; $S^{(3)} \leq_1 A_5 \leq_1 S^{(4)}$. Side results: $\{x \mid \bar{W}_x$ is finite$\} \equiv \overline{S^{(3)}}$; $\{x \mid W_x$ is infinite$\} \equiv \{x \mid \phi_x$ is a recursive permutation$\} \equiv \overline{S^{(2)}}$. In a footnote added in proof it is stated that $\bar{A}_6 \equiv S^{(4)}$ follows from results of Kreisel, Shoenfield and Wang [Arch. Math. Logik Grundlagenforsch. **5** (1960), 42–64; MR **22** #6709]. Methods of proof: Upper bounds for degrees are obtained using methods of A. Tarski [Fund. Math. **17** (1931), 210–239] and Kuratowski. Lower bounds are found by showing that certain naturally defined sets are 1-1 reducible to the sets under consideration. One of the techniques employed in the proof of $S^{(3)} \leq_1 A_5 \leq_1 S^{(4)}$ is that used by R. Friedberg in his solution of Post's problem [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 236–238; MR **18**, 867]. This solution of Post's problem ($\bar{A}_1 \neq A_5$) is also a corollary of the author's main results.

J. C. E. Dekker (New Brunswick, N.J.)

5572:

Markov, A. A. The theory of algorithms. Amer. Math. Soc. Transl. (2) **15** (1960), 1–14.

Translation of Trudy Mat. Inst. Steklov. **38** (1951), 176–189 [MR **13**, 811].

5573:

Huzino, Seiiti. On some sequential equations. Mem. Fac. Sci. Kyushu Univ. Ser. A **14** (1960), 50–62.

Continuing his study of sequential machines [same Mem. **12** (1958), 136–179; **13** (1959), 53–83; MR **20** #7599, 7600; **21** #4917, 4918] the author introduces operators α, too complicated to describe here, which map one machine into another. He then discusses when the equation $\alpha M = N$ has solutions M.

S. Ginsburg (Santa Monica, Calif.)

5574:

Dedekind, Richard. ★Stetigkeit und irrationale Zahlen. 6te unveränderte Aufl. Friedr. Vieweg & Sohn. Braunschweig, 1960. 22 pp. DM 2.40.

Companion republication (original: 1872) to the same publisher's *Was sind und was sollen die Zahlen?* [8th ed., 1960; MR **21** #5576].

SET THEORY

5575:

Halmos, Paul R. ★**Naive set theory.** The University Series in Undergraduate Mathematics. D. Van Nostrand Co., Princeton, N.J.-Toronto-London-New York, 1960. vii + 104 pp. $3.50.

First of all, the word "naive", in the title of this very concise and lucid introduction to set theory, disagrees with its common meaning (naive set theory = the pre-axiomatic, Cantorian, absolute one). Perhaps the adjective "naive" is due to the author's modesty; in fact, the textbook is built on a didactically excellent, informal but exact exposition of the Zermelo-Fraenkel-Skolem axioms and of their standard consequences ("... axiomatic set theory from the naive point of view", as the author says in the preface).

In an almost conversational though not uncritical style, the author quickly proceeds from basic facts to more involved ones, by means of a great variety of instructive examples. If one disregards problems of the foundations and needs only "some" set theory (as a basic standard tool only), this seems to be the best mode. The specific manner of exposition of the standard material may be seen from the contents (as completed by the reviewer's notes added in parentheses): Preface. 1: The Axiom of Extension. 2: The Axiom of Specification (i.e., the Aussonderungsaxiom, with an instructive solution of the Russel paradox; it is not stated that this axiom can be dispensed with on account of the later Axiom of Substitution). 3: Unordered pairs (and the corresponding axiom). 4: Unions and intersections (with the Axiom of Unions for any collection of sets). 5: Complements and powers (of course, relative complements, with a mention of Boolean algebras of sets and with a handling of intersections of empty collections of sets; the Axiom of Power Set requires an overset of the power set in question). 6: Ordered pairs (and Cartesian products of two factors, in the Wiener-Kuratowski sense, including some interesting remarks against the objection of artificiality of this notion). 7: Relations (including the discussion of equivalence relations). 8: Functions (i.e., mappings which are sets; this section ends with the definitions $0 = \emptyset$, $1 = \{\emptyset\}$, $2 = \{\emptyset, \{\emptyset\}\}$ of 0, 1, 2 and with the definition of the set 2^X of all characteristic functions of subsets of a set X). 9: Families (of indexed sets, in connection with general Cartesian products). {In the reviewer's opinion, here on p. 36 is the only unsatisfactory place of the whole book. According to common license, the author writes (line 11): "The difference between Z (= the set of all families z, indexed by a certain $\{a, b\}$ with $a \neq b$, such that $z_a \in X$ and $z_b \in Y$) and $X \times Y$ is merely a matter of notation." Disregarding subtleties concerning the difference between mathematical and metamathematical iteration of the Cartesian product, the reader can be worried in observing that $X \times (Y \times Z^*) = (X \times Y) \times Z^*$ is not true in general (by § 6) whereas the mentioned replacement of $X \times Y$ by Z yields an associative "Cartesian product".} 10: Inverses and composites (of functions; the generalization to relations is mentioned). 11: Numbers (i.e., naturals; since the Axiom of Infinity is formulated as the existence of sets closed with respect to the unary successor operation $x^+ = x \cup \{x\}$, the set ω of (set-theoretical) naturals can be defined as the minimal set containing $\emptyset$ and closed with respect to the successor operation). 12: The Peano axioms (inferring them from § 11 and including the recursion theorem on definition by induction). 13: Arithmetic (basic facts of the arithmetic of set-theoretical naturals). 14: Order (partial and total; basic facts). 15: The Axiom of Choice (in the formulation: The Cartesian product of a non-empty family of non-empty sets is non-empty). 16: Zorn's Lemma (with the proof of Zermelo). 17: Well ordering (with the well ordering theorem proved by Zorn's lemma). 18: Transfinite recursion (with the theorem on definition by transfinite induction in any well ordered set). 19: Ordinal numbers (starting with the Axiom of Substitution; previously ascribing, by transfinite recursion, to any given well ordered set a certain basic set well ordered by the $\in$-relation (so that two similar well ordered sets obtain the same $\in$-well ordered set), the author thus introduces v. Neumann's ordinals in a "constructive way"). 20: Sets of ordinal numbers (defining v. Neumann's ordinals as usual and proving the counting theorem explicitly). 21: Ordinal arithmetic (i.e., addition, multiplication and exponentiation of ordinals). 22: The Schröder-Bernstein Theorem. 23: Countable sets (with Cantor's theorem on the power set; set operations preserving and not preserving countability are discussed). 24: Cardinal arithmetic (addition, multiplication, exponentiation of cardinals, leaving open which sets they are). The final § 25 deals with cardinal numbers (as initial ordinals; a brief mention is made of the Special and the General Continuum Hypothesis).

As to further specifications, let us add the following notes. "Urelemente" (= individuals which are not sets) are excluded (cf. p. 1). No mention is made concerning the axiom of regularity (= the "Fundierungsaxiom"). No explicit distinction is made between an axiom and an axiom schema.

The reviewer has observed only one notable misprint: on p. 25, line 15, instead of "$a = b$ and $x = y$" read "$a = x$ and $b = y$".

Concluding, the reviewer wants to say that the book is to be recommended as one of the best modern introductory text books of set theory. *L. Rieger* (Prague)

5576:

Suppes, Patrick. ★**Axiomatic set theory.** The University Series in Undergraduate Mathematics. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960. xii + 265 pp.

This is one of the few modern and carefully written though very readable text books on the axiomatic set theory of Zermelo, Fraenkel and Skolem (Z. F. S.). As to the material, the book is a self-contained extension, completion and more advanced elaboration of the minimum of the subject covered by the (above reviewed) associated but more elementary textbook of Halmos. As to the style, the present book is quite specific, of course. The main additions and deviations (in comparison with the former book; cf. the foregoing review) seem to be as follows.

(i) The logical tools (only mentioned by Halmos) are developed and used explicitly, in elaborated metamathematical formulations of axiom and theorem schemas, though their use is half-formal, as it obviously must be, applying the symbolic first order language of set theory rather than the formal first order predicate calculus itself. (ii) Individuals that are not sets ("Urelemente") are admitted, causing some deviations in axioms and axiom

schemas. Thus cardinals could be introduced as special individuals by a special axiom (ascribed to Tarski). (iii) Finite sets in the sense of Tarski's definition (a set is finite if and only if every non-empty family of subsets of the given set has a member minimal in the sense of inclusion) are treated separately; finite cardinals are well ordered (as cardinals of finite sets). (iv) Finite ordinals and Peano arithmetic are handled much more completely than Halmos does; moreover, basic facts about rationals and reals (via Cauchy sequences) are derived. (Finite ordinals are v. Neumann's ordinals.) (v) The axiom of regularity is assumed and briefly discussed. (vi) The introduction of the axiom of choice is postponed so far as possible. (vii) Inaccessible cardinals are mentioned (at the end of the book). (viii) A great and (didactically) well ordered variety of exercises includes much further material.

By these eight points, the book comes near to the excellent Polish monograph of Kuratowski and Mostowski [*Teoria mnogości*, Polskie Towarzystwo Mat., Warsaw-Wrocław, 1952; MR **14**, 960]; but there, in contrast to the present book, also ordinals are treated like cardinals.

The author proceeds in a didactically very natural way. Starting with a brief discussion of classical paradoxes, especially with the set-theoretical ones, he presents the main motives for axiomatizing set theory; then he prepares the introduction of one axiom or axiom schema after another by showing its need, as well as by summarising historical development of the idea of the axiom in question. This style makes the book not only very instructive for the student but also interesting for the specialist.

Many deep results concerning questions of consistency and independence of axioms are at least mentioned. The references (in contrast to Halmos book, where there are no references at all) are quite extensive; the only criticism that the reviewer feels compelled to make is their somewhat subjective choice. Thus, we find in the book a number of names of less known authors but not, e.g., that of Mostowski (despite, e.g., his important and well-known result that the axiom of choice is independent of the ordering principle, of course disregarding the axiom of regularity). In any case, the question of independence for the axiom of choice could be mentioned more explicitly. Further, there is, e.g., no mention of the famous Souslin problem, though in the reviewer's opinion every advanced student should know of it. On the other hand, note that the author calls attention to the differences between the Z. F. S. system and the system of v. Neumann, Bernays and Gödel, by several valuable comments, though perhaps they could be more complete.

Unfortunately, we find a certain number of minor omissions and misprints; concerning them the reviewer refers to the already issued Errata, to be corrected in the second printing, 1961, of the book.

These objections, even if admitted, cannot make the book less valuable. *L. Rieger* (Prague)

5577:

Ohkuma, Tadashi. On some relations concerning the operations P_α and S_α on classes of sets. J. Math. Soc. Japan **11** (1959), 177–195.

Let α, β, γ, δ, ε be any ordinal numbers. For any class K of sets let P_α [resp. S_α] denote the class of all the intersections [unions] of subclasses of K of power $< \aleph_\alpha$. Consequently, P_α, S_α are functions; therefore the compound functions (called "polynomials") $P_\alpha S_\beta$, $S_\beta P_\alpha$, $P_\alpha S_\beta P_\gamma, \cdots$ are defined in an obvious way. The polynomials are ordered so that $P \leqq Q$ means $P(K) \subseteq Q(K)$ for every class K. The paper deals particularly with the study of following relations: (1) $S_\alpha P_\beta S_\gamma \leqq P_\delta S_\varepsilon$ (dually: $P_\alpha S_\beta P_\gamma \leqq S_\delta P_\varepsilon$), (2) $S_\alpha P_\beta S_\gamma \leqq S_\delta P_\varepsilon$, (3) $P_\delta S_\varepsilon \leqq P_\alpha S_\beta P_\gamma$, (4) $P_\delta S_\varepsilon \leqq S_\alpha P_\beta S_\gamma$. Let $\pi_\alpha(\beta)$ denote the least ordinal γ such that $\prod_{\sigma<\alpha'} m_\sigma < \aleph_\gamma$ for every $\alpha' < \alpha$ and every α'-sequence m_σ of cardinal numbers $< \aleph_\beta$. Let $p(\beta)$ be the least ordinal γ satisfying $\aleph_\beta^{\aleph_\gamma} > \aleph_\beta$. Let $q(\beta)$ be the least ordinal γ satisfying $m^{\aleph_\gamma} \geqq \aleph_\beta$ for some $m < \aleph_\beta$.

Theorem 2: (1) $\Leftrightarrow \pi_\alpha(\beta) \leqq \delta$ and: if $\mathrm{cf}\,(\gamma) < \alpha \leqq \gamma + 1$ then $\gamma < \varepsilon$, otherwise $\max\{\alpha, \gamma\} \leqq \varepsilon$.

Theorem 3: (2) $\Leftrightarrow \beta \leqq \varepsilon$ and: if $\mathrm{cf}\,\pi_\beta(\gamma) < \alpha \leqq \pi_\beta(\gamma) + 1$ then $\pi_\beta(\gamma) < \delta$, otherwise $\max(\pi_\beta(\gamma), \alpha) \leqq \delta$.

Let $(3)_1$, $(3)_2$ mean the left- and right-hand sides of (3) respectively.

Theorem 5: $(3)_1 = (3)_2 \Leftrightarrow$

$$[\alpha = \delta \wedge \beta = \varepsilon \wedge \pi_\beta(\gamma) < \delta]$$
$$\vee [\beta = \varepsilon \wedge \alpha = \delta = \mathrm{cf}\,\alpha = \pi_\beta(\gamma)]$$
$$\vee [\alpha < \beta = \gamma = \delta = \varepsilon = \pi_\beta(\beta)].$$

Theorem 7: $(4)_1 = (4)_2 \Leftrightarrow$

$$[\delta = \beta \wedge \varepsilon = \gamma \wedge \pi_\beta(\beta) = \beta \wedge \alpha \leqq \mathrm{cf}\,\gamma]$$
$$\vee [\gamma < \alpha = \beta = \delta = \varepsilon = \pi_\delta(\alpha)]$$
$$\vee [\mathrm{cf}\,\gamma < \alpha = \beta = \delta = \pi_\alpha(\alpha) \wedge \varepsilon = \gamma + 1].$$

Theorem 8:

$$\pi_\alpha(\beta) = \beta \Leftrightarrow \alpha \leqq \min\{\mathrm{cf}\,\beta, q(\beta)\};$$
$$\pi_\alpha(\beta) = \beta + 1 \Leftrightarrow q(\beta) < \alpha \leqq p(\beta);$$
$$\pi_\alpha(\beta) \geqq \beta + 2 \Leftrightarrow \alpha > \mathrm{cf}\,\beta \vee \alpha > \max\{p(\beta), q(\beta)\}.$$

The paper is connected with papers by W. Sierpiński and A. Tarski, Fund. Math. **15** (1930), 292–300; A. Koźniewski and A. Lindenbaum, ibid. **15** (1930), 342–355; A. Tarski, ibid. **16** (1930), 181–304. Cf. also the reviewer's papers, Pacific J. Math. **7** (1957), 1125–1143; Rad Jugoslav. Akad. Znan. Umjet. Odjel. Mat. Fiz. Tehn. Nauke **4** (**292**) (1953), 233–249; Bull. Internat. Acad. Yougoslave Cl. Sci. Math. Phys. Tech. **5** (1955), 97–107 [MR **20** #9, #4498]. *Đ. Kurepa* (Zagreb)

COMBINATORIAL ANALYSIS

See also 5961.

5578:

Hoffman, Alan J. Some recent applications of the theory of linear inequalities to extremal combinatorial analysis. Proc. Sympos. Appl. Math., Vol. 10, pp. 113–127. American Mathematical Society, Providence, R.I., 1960.

It has been discovered by a number of writers and the author in particular that certain types of combinatorial problems can be solved using the duality properties of systems of linear inequalities. The prototype of these combinatorial problems is the celebrated theorem on systems of distinct representatives of P. Hall. In this paper the author (1) shows how to prove Hall's theorem via inequalities, (2) mentions various other theorems on systems of representatives and flows in networks which can be proved by these methods, (3) shows how the results of (2) may be obtained directly from Hall's theorem,

(4) states what he calls the "most general" (the quotation marks are his) theorem of the Hall type concerned with the following problem: Given a family of subsets $S_1, \cdots, S_m$ of a finite set S, find a measure on S so that the measure of the set S_i shall lie between prescribed bounds a_i and b_i for each i. A certain set of inequalities are given which are necessary in order that this problem have a solution, and it is asserted that these inequalities are also sufficient if, and only if, the incidence matrix of the sets S_i has the unimodular property, meaning that all of its submatrices have determinant 0, 1 or -1. Further remarks are made about conditions under which incidence matrices are unimodular, and some lines for further research are mentioned. *D. Gale* (Providence, R.I.)

5579:

Tucker, A. W. A combinatorial equivalence of matrices. Proc. Sympos. Appl. Math., Vol. 10, pp. 129–140. American Mathematical Society, Providence, R.I., 1960.

Two $m \times n$ matrices A, B are said to be combinatorially equivalent if

$$[I_m, A]P\begin{bmatrix} -B \\ I_n \end{bmatrix} = 0,$$

where P is a permutation matrix. This defines an equivalence relation, the equivalence classes of which contain at most $(m+n)!$ matrices each. It is proved that A, B belong to the same class if and only if A is obtainable from B by a finite succession of operations of the three following types: (i) interchange of two rows; (ii) interchange of two columns; (iii) 'pivotal transformations' of order one. *D. E. Rutherford* (St. Andrews)

5580:

Priday, C. J. On Langford's problem. I. Math. Gaz. **43** (1959), 250–253.

The problem is to arrange the numbers 1, 1, 2, 2, $\cdots$, n, n in a sequence in such a way that for $r = 1, 2, \cdots, n$, the two r's are separated by exactly r places: for example, 41312432. Such a sequence is called perfect. If a sequence satisfies the same condition but has a gap one place from one end, it is called hooked; for example, 121 * 2. The author shows that for every n there exists either a perfect sequence or a hooked sequence. He conjectures that the perfect sequence exists precisely when $n = 4m$ or $4m-1$, and that the hooked sequence exists precisely when $n = 4m-2$ or $4m-3$. [See the following review.] *N. J. Fine* (Philadelphia, Pa.)

5581:

Davies, Roy O. On Langford's problem. II. Math. Gaz. **43** (1959), 253–255.

The author proves Priday's conjecture [see preceding review], and relates this problem to two similar ones of Skolem. *N. J. Fine* (Philadelphia, Pa.)

5582:

Andrews, W. S. ★Magic squares and cubes. With chapters by other writers. 2nd ed., revised and enlarged. Dover Publications, Inc., New York, 1960. viii+419 pp. $1.85.

Unaltered republication of 2nd edition [Open Court, La Salle, Ill., 1917]. A compilation of articles by Andrews, Paul Carus, L. S. Frierson, C. A. Browne, H. M. Kingery, H. A. Sayles, John Worthington, D. F. Savage, C. Planck, F. A. Woodruff. In addition to magic squares (plain and ornate) and cubes, there are morsels on magic circles, spheres, stars and octahedroids.

5583:

Nikolai, Paul J. Permanents of incidence matrices. Math. Comput. **14** (1960), 262–266.

The incidence matrices in question are those associated with v, k, λ configurations; square matrices of size v of zeroes and ones, each row with exactly k ones, each pair of rows with exactly λ ones in common. The permanent of an incidence matrix is the number of systems of distinct representatives of the corresponding configuration. Numerical exploration of the five nonisomorphic 15, 7, 3 configurations determined by H. K. Nandi [Sankhyā **7** (1946), 313–316; MR **8**, 127] shows different permanents for all but two. Also the permanents of all cyclic configurations with $v < 23$ have been computed. The computation is by means of a theorem due to H. J. Ryser expressing the permanent in terms of products of row sums of a succession of matrices obtained by inclusion and exclusion. *J. Riordan* (New York)

5584:

Raney, George N. Functional composition patterns and power series reversion. Trans. Amer. Math. Soc. **94** (1960), 441–451.

Let $a_1, a_2, \cdots$ be an infinite sequence of natural numbers 0, 1, 2, $\cdots$, such that $\sum a_i$ is finite. The author defines the numbers $L = L(n; a_1, a_2, \cdots)$ combinatorially and shows that

$$L(n; a_1, a_2, \cdots) = \Big(\sum_{i=0}^{\infty} a_i\Big)!n \Big/ \prod_{i=0}^{\infty} (a_i!)m,$$

where $m = n + \sum_{i=1}^{\infty} ia_i$, $a_0 = n + \sum_{i=1}^{\infty} (i-1)a_i$, and $L = 1$ if $m = n = 0$. He then derives some identities involving the numbers L, and uses them to prove a Lagrange inversion formula on formal power series and a convolution formula given by H. W. Gould [Amer. Math. Monthly **64** (1957), 409–415; MR **19**, 379]. *Rimhak Ree* (Vancouver, B.C.)

5585:

Gould, H. W. Generalization of a theorem of Jensen concerning convolutions. Duke Math. J. **27** (1960), 71–76.

Jensen [Acta Math. **26** (1902), 307–318] has proved the identity

$$\sum_{k=0}^{n} \binom{\alpha+\beta k}{k}\binom{\gamma-\beta k}{n-k} = \sum_{k=0}^{n} \binom{\alpha+\gamma-k}{n-k}\beta^k.$$

In the present paper the author obtains the following "Abel-type" analog:

$$(1) \qquad \sum_{k=0}^{n} \frac{(\alpha+\beta k)^k}{k!}\frac{(\gamma-\beta k)^{n-k}}{(n-k)!} = \sum_{k=0}^{n} \frac{(\alpha+\gamma)^k}{k!}\beta^{n-k}.$$

The proof depends on the recurrence

$$G(\alpha, \gamma, n) - \beta G(\alpha+\beta, \gamma-\beta, n-1) = (\alpha+\gamma)^n/n!,$$

where $G(\alpha, \gamma, n)$ stands for the left member of (1). More

generally the author obtains various necessary and sufficient conditions that the sum

$$S(\alpha, \gamma, n) = \sum_{k=0}^{n} G_k(\alpha, \beta) G_{n-k}(\gamma, \beta)$$

satisfy the recurrence

$$S(\alpha, \gamma, n) - \beta S(\alpha+\varepsilon, \gamma, n-1) = G_n(\alpha+\gamma, \beta).$$

L. Carlitz (Durham, N.C.)

5586:

Gould, H. W. Stirling number representation problems. Proc. Amer. Math. Soc. **11** (1960), 447–451.

Put

$$\prod_{k=0}^{n} (1+kx) = \sum_{k=0}^{\infty} S_1(n, k) x^k,$$

$$\prod_{k=0}^{n} (1-kx)^{-1} = \sum_{k=0}^{\infty} S_2(n, k) x^k.$$

Schlömilch [J. Reine Angew. Math. **44** (1852), 344–355] proved a formula equivalent to

$$(1) \qquad S_1(n-1, k) = \sum_{j=0}^{k} \binom{k+n}{k-j} \binom{k-n}{k+j} S_2(j, k),$$

due to Schläfli [J. Reine Angew. Math. **67** (1867), 179–182]. In the present paper a number of formulas similar to (1) are obtained. In particular the formula

$$S_2(n-k, k) = \sum_{j=0}^{k} \binom{k-n}{k+j} \binom{k+n}{k-j} S_1(k+j-1, k)$$

is a companion to (1). We note also the following pair of formulas:

$$S_1(n-1, k) = \sum_{t=0}^{k} K(t) S_1(k+t-1, k),$$

$$S_2(n-k, k) = \sum_{t=0}^{k} K(t) S_2(t, k),$$

where

$$K(t) = \sum_{j=0}^{k} \binom{k-n}{k+j} \binom{k+n}{k-j} \binom{2k+j}{k-t} \binom{-j}{k+t}.$$

The proofs make use of properties of Bernoulli numbers of higher order. *L. Carlitz* (Durham, N.C.)

5587:

Carlitz, L. Note on Nörlund's polynomial $B_n^{(z)}$. Proc. Amer. Math. Soc. **11** (1960), 452–455.

H. W. Gould [preceding review] has proved that

$$(1) \qquad B_k^{(z)} = \sum_{j=0}^{k} (-1)^j \binom{k+1}{j+1} B_k^{(-jz)},$$

$$(2) \qquad (-1)^k \binom{z}{k} B_k^{(k-z)} = \sum_{j=0}^{k} \binom{k-z}{k+j} \binom{k+z}{k-j} \binom{k+j-1}{k} B_k^{(j+k)},$$

where the $B_k^{(z)}$ $(k=0, 1, 2, \cdots)$ are the Nörlund polynomials, defined by

$$\left(\frac{x}{e^x-1}\right)^z = \sum_{k=0}^{\infty} B_k^{(z)} \frac{x^k}{k!}.$$

The author here presents brief new proofs of both identities, and, more generally, observes that his arguments apply equally well to establish (1) and (2) with $B_k^{(z)}$ replaced by any polynomial $f(z)$, of degree not exceeding k, which satisfies $f(0)=0$ (for $k=0$, even this last condition is not needed). The formula at the top of p. 454 is incorrectly stated, but does not essentially affect the argument.

{Reviewer's comment. Another way of looking at (2) is as follows: On writing $\binom{z}{k} f(k-z) = g(z)$, we have $g(-j) = (-1)^k \binom{j+k-1}{k} f(k+j)$, so that the author's generalization of (2) is equivalent to $g(z) = \sum_{j=0}^{k} \binom{k-z}{k+j} \binom{k+z}{k-j} g(-j)$, valid for any polynomial $g(z)$, of degree not exceeding $2k$, subject also (for $k \geq 1$) to the condition $g(i)=0$ $(i=0, 1, \cdots, k)$. To prove this polynomial identity (of degree at most $2k$ in z) it suffices to check it at the $2k+1$ integral points in the range $-k \leq z \leq k$; but, for these z, $\binom{k-z}{k+j} \binom{k+z}{k-j} = 0$ unless $j=-z$, so that in each case the sum reduces to at most one term, whence the verification is easily completed. Indeed, the hypothesis $g(0)=0$ is not really needed, so we have a strict generalization.}

M. P. Drazin (Baltimore, Md.)

5588:

Gould, H. W. The Lagrange interpolation formula and Stirling numbers. Proc. Amer. Math. Soc. **11** (1960), 421–425.

Let $f_k(z)$ denote an arbitrary polynomial of degree k such that $f_k(0)=0$ for $k \geq 1$. Put

$$(-1)^k F_1(n-1, k) = \binom{n-1}{k} f_k(n),$$

$$F_2(n, k) = \binom{n+k}{k} f_k(-n).$$

The reviewer [#5587 above] showed that

$$(1) \qquad F_1(n-1, k) = \sum_{j=0}^{k} \binom{k-n}{k+j} \binom{k+n}{k-j} F_2(j, n),$$

$$(2) \qquad F_2(n-k, k) = \sum_{j=0}^{k} \binom{k-n}{k+j} \binom{k+n}{k-j} F_1(k+j-1, k),$$

thus generalizing some formulas for the Stirling numbers of the first and second kind [see review #5586]. In the present paper the author shows that (1) and (2) can be proved by means of the Lagrange interpolation formula.

L. Carlitz (Durham, N.C.)

5589:

Ghouila-Houri, Alain. Un résultat relatif à la notion de diamètre. C. R. Acad. Sci. Paris **250** (1960), 4254–4256.

A directed graph (digraph) is strongly connected (strong) if any two points are mutually reachable along directed paths. The diameter of a strong digraph is the maximum value of $d(u, v)$, the distance from any point u to any other point v. Theorem: Among all strong digraphs with n points and m directed lines, the maximum diameter is $n-1$ when $n \leq m \leq \frac{1}{2}n(n+1)-3$, and is

$$[n+1/2-\sqrt{(2m-n^2-n+17/4)}]$$

when $\frac{1}{2}n(n+1)-3 < m \leq n(n-1)$.

F. Harary (Ann Arbor, Mich.)

5590:

Ghouila-Houri, Alain. Une condition suffisante d'existence d'un circuit hamiltonien. C. R. Acad. Sci. Paris **251** (1960), 495–497.

Let a finite directed graph with X ($\geqq 2$) vertices have the following two properties. (i) If x and y are any two distinct vertices, then the graph contains a directed path from x to y and a directed path from y to x. (ii) If x is any vertex, then the sum of the number of vertices $\neq x$ with which x is joined by edges directed away from x and the number of vertices $\neq x$ with which x is joined by edges directed towards x is at least X. Then the graph contains a directed circuit to which all vertices of the graph belong.

G. A. Dirac (Hamburg)

5591:

Erdős, P.; Gallai, T. On maximal paths and circuits of graphs. Acta Math. Acad. Sci. Hungar. **10** (1959), 337–356. (Russian summary, unbound insert)

The authors consider a problem of P. Turán: for a given number of vertices, what is the minimum number of edges that ensures that a graph shall have a subgraph of a specified kind? They study the cases in which the specified subgraph is an arc, a polygon (circuit), or a regular graph of valency (degree) 1. In each case estimates are obtained for the minimum number of edges, and in some isolated cases the number is determined exactly. The authors single out the following theorems for special mention. (1) Every graph with n vertices and more than $\frac{1}{2}l(n-1)$ edges ($l\geqq 2$) contains a polygon with more than l edges. The value $\frac{1}{2}l(n-1)$ is the best possible if and only if $n=q(l-1)+1$. (2) For all $n\geqq \frac{1}{2}(k+1)^3$, $k\geqq 1$, every graph with n vertices and more than $nk-\frac{1}{2}k(k+1)$ edges contains an arc or polygon with more than $2k$ edges. The value $nk-\frac{1}{2}k(k+1)$ is the best possible.

There is another kind of problem, studied by Zarankiewicz and Dirac, in which the existence of specified subgraphs is asserted if the valencies are sufficiently high. Several theorems of this type are proved in § 1, and are used later in the proofs of the "Turánian" theorems.

W. T. Tutte (Toronto)

5592:

Dirac, G. A. Paths and circuits in graphs: extreme cases. Acta Math. Acad. Sci. Hungar. **10** (1959), 357–362. (Russian summary, unbound insert)

The author gives a complete characterization of the graphs in which the valency of each vertex is $\geqq d\geqq 2$ and which contain no arc of length $>2d$. He remarks that his paper can be regarded as a supplement to the paper of P. Erdős and T. Gallai [preceding review].

W. T. Tutte (Toronto)

5593:

Tutte, W. T. Convex representations of graphs. Proc. London Math. Soc. (3) **10** (1960), 304–320.

The present paper gives combinatorial conditions on a graph G which are necessary and sufficient that it have a convex representation; that is, that there exist a mapping h of the vertices of G onto points in a plane and of the edges of G into the straight line segments joining the appropriate vertices such that each component of the complement of $h(G)$ is the interior or exterior of a convex polygon. The usual characterization of planar graphs by the absence of certain subgraphs is not used; instead the conditions listed below are together necessary and sufficient for the existence of a convex representation of G.

(1) G is simple; that is, no edge is a loop from one vertex to the same vertex and no two edges share a pair of vertices. (2) G is not separable; that is, no two subgraphs of G exist whose union is G and whose intersection is a single vertex of G. (3) G has a planar mesh; that is, there exist in G a set of k elementary cycles $C_1, \cdots, C_k$ (which in the representation correspond to the boundary paths around the convex polygons) such that (a) every edge of G is an edge of precisely two C_i, and (b) every cycle in G is the sum (mod 2) of an appropriate set of the C_i. (4) C_1 is three-linked to every elementary cycle C in G which is disjoint from C_1; that is, if S and S' are complementary sets of edges of G such that $S\supseteq C_1$ and $S'\supseteq C$, then at least three vertices of G are ends of edges both from S and from S'. (This C_1 corresponds to the outermost convex polygon in the representation.) (5) No branch of G spanning C_1 ends at the vertices of a single edge of C_1; that is, if a path in G starts from a vertex p of C_1 and proceeds along edges not in C_1 through vertices which have precisely two edges each until it reaches another vertex q of C_1, then p and q are not the ends of one edge of C_1.

M. M. Day (Urbana, Ill.)

5594:

Read, R. C. The number of k-coloured graphs on labelled nodes. Canad. J. Math. **12** (1960), 410–414.

$F_n(k)$ denotes the total number of k-coloured graphs with n labelled nodes, $f_n(k)$ denotes the total number of connected k-coloured graphs with n labelled nodes and $M_n(k)$ denotes the total number of graphs with n labelled nodes coloured with at most k colours. The following results are established.

$$F_n(k)=\sum_{r=1}^{n-1}\binom{n}{r}2^{r(n-r)}F_r(k-1).$$

$$M_n(k)=\sum_{r=0}^{n}\binom{n}{r}2^{r(n-r)}M_r(k-1)\quad\text{with } M_0(k)=1.$$

$$f_n(k)=F_n(k)-\sum_{r=1}^{n-1}\binom{n-1}{r-1}F_{n-r}(k)f_r(k).$$

G. A. Dirac (Hamburg)

ORDER, LATTICES

See also 5602, 5673, B6097.

5595:

Hostinsky, L. Aileen. Splitting criteria for modular lattices. Proc. Amer. Math. Soc. **11** (1960), 23–28.

Verf. führt ihre Untersuchungen [Duke Math. J. **18** (1951), 331–342; Amer. J. Math. **73** (1951), 741–755; MR **12**, 795; **13**, 525] über vollständige modulare Verbände L, die der Bedingung $a\sum_\alpha p_\alpha=\sum_\alpha ap_\alpha$ für aufsteigende Ketten p_α genügen, fort. Unter den Homomorphismen der Quotienten p/p' von L auf Quotienten q/q' werden die "uniformly splitting" Endomorphismen (durch eine gewisse direkte Zerlegbarkeit der Elemente von p/p') ausgezeichnet. Mit den hier entwickelten

Kriterien für "uniform splitting" erhält Verf. verbandstheoretische Verallgemeinerungen gruppentheoretischer Kriterien für die Existenz normaler Untergruppen als direkte Faktoren [vgl. R. Baer und C. Williams, Bull. Amer. Math. Soc. **55** (1949), 729–743; MR **11**, 321].
P. Lorenzen (Kiel)

5596:

Onicescu, Octav. Notes sur les *b*-algèbres. An. Univ. "C. I. Parhon" București. Ser. Ști. Nat. No. 22 (1959), 17–22. (Romanian. Russian and French summaries)

Author's summary: "Dans la première note, l'auteur met en evidence une décomposition canonique de chaque algèbre de Boole affectée d'une mesure finie, en un ensemble dénombrable d'atomes, en un ensemble dénombrable de classes de quasiatomes, et une algèbre de Boole sans atomes et quasiatomes, appelée le noyau continu. Dans la seconde note, l'auteur démontre le théorème suivant: toute partie de l'algèbre affectée d'une mesure finie admet un élément maximal. Dans la troisième note l'auteur construit un sous-espace de l'espace des paires d'éléments de l'algèbre Ω, dont un corps de parties est isomorphe à Ω." Cf. Rev. Math. Pures Appl. **4** (1959), 345–350 [MR **22** #6743].

5597:

Constantinescu, Paul. Sur la classification des fonctions booléennes symétriques. Acad. R. P. Romîne. Stud. Cerc. Mat. **11** (1960), 193–206. (Romanian. Russian and French summaries)

For $n \leq 5$, the author, using binomial coefficients, counts the number t_n of symmetry types of Boolean functions of n variables such that one of the functions of that symmetry type is symmetric in all its variables. He explicitly gives a function of each such symmetry type, and gives the total number T_n of functions of all these types. His results are summarized by the triples (n, t_n, T_n), which are (1, 3, 4), (2, 5, 12), (3, 10, 52), (4, 19, 228), and (5, 36, 964).
E. F. Moore (Murray Hill, N.J.)

5598:

Sagalovič, Yu. L. On group invariance of Boolean functions. Uspehi Mat. Nauk **14** (1959), no. 6 (90), 191–195. (Russian)

Previously Pólya [J. Symb. Logic **5** (1940), 98–103; MR **2**, 65], Slepian [Canad. J. Math. **5** (1953), 185–193; MR **15**, 93] and McCluskey [Bell System Tech. J. **35** (1956), 1445–1453; MR **18**, 624] considered the problem of the symmetry classes of Boolean functions in n variables. The methods used are subject to awkwardness even for values of n not appreciably large. The paper under review returns to the problem and presents further quantitative results, too elaborate to be restated here, relevant to the substitution groups involved. A partial table of results (for $n \leq 5$) is included.
R. M. Baer (Berkeley, Calif.)

5599:

Sagalovič, Yu. L. Number of symmetry types among (1, *k*) terminal contacts. Dokl. Akad. Nauk SSSR **130** (1960), 72–73 (Russian); translated as Soviet Physics. Dokl. **5**, 29–30.

This paper extends D. Slepian's work on the number of symmetry types of Boolean functions of n variables [Canad. J. Math. **5** (1953), 185–193; MR **15**, 93] from (1, 1) terminal contacts to (1, k) terminal contacts. The number of symmetry types is computed for $n, k = 1, 2, 3, 4$; for example, when $n = k = 4$, there are 2001547791980875 symmetry types.
J. Hartmanis (Schenectady, N.Y.)

5600:

Kuznecov, A. V. A property of functions realized by non-planar non-repeating networks. Trudy Mat. Inst. Steklov. **51** (1958), 174–185. (Russian)

Given any planar two-terminal contact network S which realizes a Boolean function Φ, one can by a well-known algorithm construct a planar network S which has the same number of contacts as S and which realizes the dual Φ^* of Φ. It is shown here that this duality principle cannot be extended to non-planar networks. In fact the following theorem is true: Let a Boolean function Φ be realized by a non-repeating strongly-connected network S; in order that Φ^* may be realized by a non-repeating network it is necessary and sufficient that S be planar. To prove necessity assume S is non-planar. By fixing the condition of some of its contacts S can be transformed into a "similar" network T which, by Kuratowski's necessary and sufficient condition that a finite graph be planar [Fund. Math. **15** (1930), 271–283], has one of three definite forms. T realizes a function G whose dual G^* can be realized by a non-repeating network if Φ^* can. It is then shown that G^* cannot be so realized.
W. J. Feeney (Weston, Mass.)

5601:

Fortet, R. L'Algèbre de Boole et ses applications en recherche opérationnelle. Cahiers Centre Études Rech. Oper. no. 4 (1959), 5–36.

This paper discusses at some length the possibility of formulating various problems in terms of Boolean algebra. For example, the problem of coloring a particular map with 4 colors can be expressed in this way. Also the problem of constructing block designs can be put into this form. The author remarks that computing machines can carry out the Boolean calculations. But, as he says, he has not investigated in detail the effectiveness of this formulation for the problems considered, except for a few fairly simple ones.
Marshall Hall, Jr. (Pasadena, Calif.)

GENERAL MATHEMATICAL SYSTEMS

5602:

Schmidt, E. Tamás. Congruence relations of algebraic structures. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **9** (1959), 163–174. (Hungarian)

The congruence relations $\Theta, \Phi, \cdots$ of a universal algebra A form a complete lattice with the partial ordering $\Theta \leq \Phi . \leftrightarrow . x \equiv y(\Theta) \rightarrow x \equiv y(\Phi)$ [G. Birkhoff, *Lattice theory*, Amer. Math. Soc., New York, 1940; MR **1**, 325]; the author denotes this lattice by $\Theta(A)$. The paper is intended as a first step towards the investigation of general algebraic structures and their congruence relation lattices.

Θ_{ab} denotes the first congruence relation for which $a \equiv b$; congruence relations which can be put in this form

are called minimal. The more congruence relations of A are minimal, the easier is the investigation of $\Theta(A)$. J. Hashimoto [Osaka Math. J. **9** (1957), 87–112; MR **19**, 935] showed that if a congruence relation is generated by a finite number of minimal congruence relations, then it is inaccessible from below. The principal theorem of the present paper is: Corresponding to any algebra A there exists an algebra T such that $\Theta(A) \cong \Theta(T)$ and all elements of $\Theta(T)$ which are inaccessible from below are minimal; if A is finite then there exists a finite algebra T with these properties.

The author states that he has been able to show that if A is a distributive lattice then there exists a distributive lattice T with the properties enumerated in the theorem, but it remains an open question whether, if A is a group [ring, lattice, ...], a group [ring, lattice, ...] T with the enumerated properties exists.

Two applications of the above theorem are given: (1) It is deduced that any finite lattice is isomorphic to a sublattice of the lattice of equivalence relations over some set. This is a special case of the theorem of P. M. Whitman [Bull. Amer. Math. Soc. **52** (1946), 507–522; MR **8**, 62] that any lattice is isomorphic to a sublattice of the lattice of equivalence relations over some set. (2) A set with one or more finitary algebraic operations, whose domain of application need not extend over the whole set, is called by the author a partial algebraic structure (or algebraic structuroid). It is shown that any finite lattice is isomorphic to the lattice of congruence relations of some finite partial algebraic structure.

G. A. Dirac (Hamburg)

5603:

Foster, Alfred L. On the imbeddability of universal algebras in relation to their identities. I. Math. Ann. **138** (1959), 219–238.

The paper is concerned with algebras $\mathfrak{A} = (A, \mathfrak{o}_1, \mathfrak{o}_2, \cdots)$ of a given species $S = (n_1, n_2, \cdots)$ and algebras $\mathfrak{A}' = (A', \mathfrak{o}_1, \mathfrak{o}_2, \cdots, \mathfrak{o}_1', \mathfrak{o}_2', \cdots)$ of an overspecies $S' = (n_1, n_2, \cdots, n_1', n_2', \cdots)$. $\mathfrak{A}$ is said to be embedded in $\mathfrak{A}'$ provided $A \subseteq A'$ and the operations $\mathfrak{o}_i$ of $\mathfrak{A}$ agree with the corresponding operations of $\mathfrak{A}'$. If, in addition, $A = A'$, then $\mathfrak{A}'$ is called an S'-conversion of $\mathfrak{A}$. $|\mathfrak{A}'|$ is the set of all identities that hold in $\mathfrak{A}'$; and if $\mathscr{S}'$ is a set of identities in the operations $\mathfrak{o}_i$, $\mathfrak{o}_i'$, then $|\mathscr{S}'|$ is the set of all those identities that are logical consequences of $\mathscr{S}'$, and $|\mathscr{S}'|_S$ is the set of all those identities in the operations $\mathfrak{o}_i$ that belong to $|\mathscr{S}'|$. $|\mathfrak{A}|$ and $|\mathscr{S}|$ are defined similarly. $\mathfrak{A}$ is called an $\mathscr{S}$-algebra provided $|\mathscr{S}| \subseteq |\mathfrak{A}|$, and an exact $\mathscr{S}$-algebra in case equality holds. If some exact $\mathscr{S}$-algebra can be embedded in [converted into] an $\mathscr{S}'$-algebra, then $\mathscr{S}$ and $\mathscr{S}'$ are said to be compatible [convertibly compatible], and if every $\mathscr{S}$-algebra can be embedded in an $\mathscr{S}'$-algebra, then $\mathscr{S}$ and $\mathscr{S}'$ are said to be universally compatible.

Clearly, if $\mathscr{S}$ and $\mathscr{S}'$ are compatible, then $|\mathscr{S}'|_S \subseteq |\mathscr{S}|$, but an example shows that this condition is not sufficient. With each $\mathscr{S}$-algebra $\mathfrak{B}$ there is associated a "free $\mathscr{S}'$-algebra generated by $\mathfrak{B}$", $\mathfrak{B}[\mathscr{S}']$, and a homomorphism of $\mathfrak{B}$ into a subalgebra $\overline{\mathfrak{B}}$ of $\mathfrak{B}[\mathscr{S}']$; and it is shown that $\mathfrak{B}$ can be embedded in an $\mathscr{S}'$-algebra if and only if this map is an isomorphism. Applied to free $\mathscr{S}$-algebras these ideas yield: $\mathscr{S}$ and $\mathscr{S}'$ are compatible if and only if every free $\mathscr{S}$-algebra can be embedded in an $\mathscr{S}'$-algebra. $\mathscr{S}$ and $\mathscr{S}'$ are convertibly compatible if and only if $\mathscr{S}$ and $\mathscr{S} \cup \mathscr{S}'$ are compatible. Sufficient conditions for universal compatibility are also given, and applications to primal and categorical algebras are considered. (A finite algebra $\mathfrak{U}$ with more than one element is said to be categorical in case every $|\mathfrak{U}|$-algebra is isomorphic to a subdirect power of $\mathfrak{U}$.)

B. Jónsson (Minneapolis, Minn.)

THEORY OF NUMBERS

See also 5557, 5629, 5630, 5842.

5604:

Adrian, P. Die Bezeichnungsweise der Bernoullischen Zahlen. Mitt. Verein. Schweiz. Versich.-Math. **59** (1959), 199–206. (French, Italian and English summaries)

A discussion of some of the more common notations for Bernoulli numbers.

L. Moser (Edmonton, Alta.)

5605:

Niven, Ivan; Zuckerman, Herbert S. ★An introduction to the theory of numbers. John Wiley & Sons, Inc., New York-London, 1960. viii+250 pp. $6.25.

The authors of this introductory textbook have succeeded in packing an astonishing amount of information into a mere 250 pages, and this with no loss of intelligibility. The first four chapters deal with congruences, quadratic reciprocity, and arithmetical functions. Chapter 5 discusses selected diophantine equations and, in particular, the representation of integers as sums of 2 or 4 squares. A brief introduction to the theory of equivalence of binary quadratic forms is also included. This is followed by a short chapter on Farey fractions and a long one on continued fractions, rational approximations, and Pell's equation. Chapter 8 covers some aspects of the elementary theory of prime numbers, and contains proofs of Chebycheff's theorem and Bertrand's postulate. The remaining three chapters are a little more advanced, both in choice of material and manner of presentation, and they can be read independently of each other. Chapter 9 is an introduction to the algebraic theory of numbers. After a brief preliminary discussion of general algebraic number fields, the authors concentrate on the possibility of unique factorization into primes of integers in quadratic fields, but do not mention ideals. Chapter 10 deals with the theory of partitions mainly from the point of view of generating functions. The identities of Euler and Jacobi are proved and certain congruence properties of the partition function are established. Finally, Chapter 11 discusses asymptotic density and Schnirelmann density of sequences of integers and gives a proof of H. B. Mann's fundamental theorem on the density of the sum of two sequences. The book is pleasantly written and easy to follow; and its usefulness is greatly enhanced by numerous problems of all degrees of sophistication scattered throughout the text.

L. Mirsky (Sheffield)

5606:

Browkin, J. Certain property of triangular numbers. Wiadom. Mat. (2) **2**, 253–255 (1959). (Polish)

The author proves that there exist infinitely many pairs of triangular numbers such that their sum and

difference are also triangular numbers. The table of all pairs of triangular numbers (t_a, t_b) with this property is given for $a < 250$ $(t_a < 125250)$.
J. W. Andrushkiw (Newark, N.J.)

5607:

Goodstein, R. L. Automorphic numbers in a general scale. Math. Gaz. **43** (1959), 270–272.

A number R is said to be automorphic in a scale n, with index r, if $R^2 \equiv R \pmod{n^r}$. The author shows that if $R^2 \equiv R \not\equiv 0 \pmod{n^r}$, a solution of $x^2 \equiv x \pmod{n^{r+1}}$ is given by $x \equiv R^v \pmod{n^{r+1}}$, where v is the greatest divisor of n which is relatively prime to R.
T. Hayashida (Kanagawa)

5608:

Tatarkiewicz, K. General criterion for divisibility of numbers. Wiadom. Mat. (2) **3**, 29–31 (1959). (Polish)

A simpler proof and generalization of the criterion of divisibility by Grünbaum [Scripta Math. **21** (1955), 204–207; MR **17**, 459] is given: If $(g, m) = 1$ and s is the length (i.e., the number of digits in the period) of the expansion of $1/m$ with respect to the base g, and if

$$N = \sum_{\nu=0}^{k} a_\nu g^{s\nu} \quad (a_\nu < g^s),$$

while $n = \sum_{\nu=0}^{k} a_\nu$, then m divides N if and only if m divides n. There is a misprint on page 31: $s(m) = s - 1$ should read $s(m) = m - 1$. *J. W. Andrushkiw* (Newark, N.J.)

5609:

Ward, Morgan. The vanishing of the homogeneous product sum on three letters. Duke Math. J. **27** (1960), 619–624.

By the homogeneous product sum $H_n = H_n(x, y, z)$ is understood the sum of all the symmetric functions of weight n. In a previous paper [same J. **26** (1959), 553–562; MR **22** #1544] the author showed that for n even the equation $H_n = 0$ has only trivial solutions. He proved also by descent that $H_3 = 0$ has only trivial solutions. In the present paper the following theorem is proved:

The diophantine equation $H_n(x, y, z) = 0$ has only trivial solutions whenever $n+2$ is a prime number > 3.

The proof is by contradiction and makes use of the cyclotomic field $R(e^{2\pi i/p})$, where $p = n+2$.
L. Carlitz (Durham, N.C.)

5610:

Cassels, J. W. S. On the equation $a^x - b^y = 1$. II. Proc. Cambridge Philos. Soc. **56** (1960), 97–103.

The equation of the title, in which a and b are regarded as known, arises as a special case of Catalan's conjecture, according to which 8 and 9 are the only two consecutive positive integers which are integral powers of smaller integers. The reviewer showed [Amer. J. Math. **74** (1952), 325–331; MR **13**, 822] that the equation of the title has at most one solution, which solution can be found explicitly if it exists, and the author showed [ibid. **75** (1953), 159–162; MR **14**, 536] that if p and q are prime, $p > q \geq 2$, and $a^p - b^q = \pm 1$ with $a > 1$, $b > 1$, then $q|a$. It is shown here that under these same conditions, also $p|b$. An account of the history of the larger conjecture is also given.

The reader should note that the letters a and b have been interchanged on p. 98, lines 2 and 4 up, p. 99, line 8, and p. 100, lines 14 and 15 up.
W. J. LeVeque (Ann Arbor, Mich.)

5611:

Carlitz, L. A special higher congruence. Elem. Math. **15** (1960), 75–76.

The author proves that if p is a prime, a necessary condition for the solvability of

$$(1) \qquad f(x) = x^{p+1} + ax^p + bx + c \equiv 0 \pmod{p^2}$$

is that $d = (a+b)^2 - 4c$ be $\equiv 0$ or a quadratic residue (mod p). If $c \not\equiv ab \pmod p$, then to each solution (mod p) of

$$(2) \qquad x^2 + (a+b)x + c \equiv 0 \pmod p$$

there corresponds a unique solution (mod p^2) of (1). If $c \equiv ab \pmod p$ the author shows that corresponding to the solution $x \equiv -b \pmod p$ of (2) there are p solutions or no solutions of (1) according as $f(-b) \equiv 0 \pmod{p^2}$ or not.
R. P. Kelisky (Yorktown Heights, N.Y.)

5612:

Malyšev, A. V. Quadratic forms in an arbitrary field. A generalization of Pall's theorem. Uspehi Mat. Nauk **15** (1960), no. 3 (93), 167–172. (Russian)

Let P be a field of characteristic different from 2, and let $f = f(x_1, \cdots, x_n)$ and $g = g(y_1, \cdots, y_m)$ be quadratic forms with matrices A and B, of ranks r and s respectively, and having entries in P. It is shown that there exists a rank q representation of g by f (i.e., a matrix T with entries in P and rank q, such that $T'AT = B$) if and only if the following conditions are satisfied: $s \leq q \leq m$ and there is a form $h = h(z_1, \cdots, z_l)$ such that f is equivalent over P to

$$g + \sum_{i \leq (q-s)-(n-r)} 2u_i v_i + h,$$

where it is supposed that the sets of variables $(y_1, \cdots, y_m)$, $(u_1, \cdots, u_k, v_1, \cdots, v_k)$ and $(z_1, \cdots, z_l)$ are disjoint. This theorem reduces to a result of G. Pall's [see Theorem 5, p. 11, of B. W. Jones' *Arithmetic theory of quadratic forms*, Math. Assoc. of America, Buffalo, N.Y., 1950; MR **12**, 244] in case $n = r$, $m = s = q$, and in case $q = m$, $s = 0$ gives conditions in order that f be an m-zero form [loc. cit., p. 50, Theorem 18].
W. J. LeVeque (Ann Arbor, Mich.)

5613:

McCarthy, P. J. Some irreducibility theorems for Bernoulli polynomials of higher order. Duke Math. J. **27** (1960), 313–318.

The reviewer [same J. **19** (1952), 475–481; MR **14**, 163] has obtained some results concerning factorization properties of the Bernoulli polynomials. In the present paper the author considers the Bernoulli polynomial $B_n^{(k)}(x)$ of degree n and order k, where k is a positive integer, defined by

$$\left(\frac{t}{e^t - 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}.$$

The following results are obtained: (1) Let p be an odd prime and let $k < p$ and $1 \leq m < p - k + 1$. Then $B_{m(p-1)}^{(k)}(x)$ is irreducible. If $k = 1$, this is true for $m = p$ also. (2) Let p be an odd prime and let $k \leq p$, $t > 0$. Then $B_{m(p-1)p^t}(x)$ is irreducible for $1 \leq m \leq p$. (3) Let p be an odd prime and let $k < p$, $1 \leq n \leq p - k + 1$; let $2m = n(p-1)$. Then

$B^{(k)}_{2m+1}(x)/(x-\frac{1}{2}k)$ has an irreducible factor of degree $\geqq 2m+1-p$. (4) For any integer $k\geqq 1$ there is an integer $T=T(k)$ such that for all $t\geqq T$, $B_{2^t}^{(k)}(x)$ is irreducible.

L. Carlitz (Durham, N.C.)

5614:

Knopp, Marvin Isadore. Determination of certain roots of unity in the theory of automorphic forms of dimension zero. Duke Math. J. **27** (1960), 497–506.

Let $G(\lambda_q)$ denote the properly discontinuous group generated by the two substitutions $S(\tau)=\tau+q$ and and $T=\tau^{-1}$, where $\lambda_q=2\cos(\pi/q)$. Let Γ denote either $G(\sqrt{2})$ or $G(\sqrt{3})$ ($q=4$ or 6), and suppose that $f(\tau)$ is an automorphic form of dimension 0 belonging to Γ. That is: $f(\tau)$ is regular for $\mathscr{I}(\tau)>0$; and for every $M\in\Gamma$, $f(M\tau)=\varepsilon(M)f(\tau)$, where $\varepsilon(M)$ is a multiplier system and $|\varepsilon(M)|=1$.

The author investigates the multiplier systems $\varepsilon(M)$ for the groups denoted by Γ, and by looking at the structure for the groups Γ, he determines explicitly all multiplier systems $\varepsilon(M)$.

R. Ayoub (University Park, Pa.)

5615:

Hornfeck, Bernhard. Zur Verteilung gewisser Primzahlpotenzprodukte. Math. Ann. **139**, 14–30 (1959).

This paper is mainly concerned with the distribution of integers of the form

$$p_1^{b_1}\cdots p_k^{b_k}, \tag{1}$$

where k is fixed, while $p_1,\cdots,p_k$ are primes and $b_1,\cdots,b_k$ are integers about which additional assumptions are made. Thus, for example, when $p_1,\cdots,p_k$ are given distinct primes and, for $1\leqq i\leqq k$, b_i belongs to the given set of integers $\mathfrak{B}_i$ with positive density β_i, then the number of integers of the form (1) which do not exceed x is asymptotic to

$$\frac{1}{k!}\prod_{i=1}^{k}\frac{\beta_i}{\log p_i}\cdot(\log x)^k.$$

This result generalizes an earlier theorem of G. Pólya [Math. Z. **1** (1918), 143–148]. Another problem considered is the distribution of the numbers (1) when $p_1,\cdots,p_k$ are distinct primes belonging to a given set and $(b_1,\cdots,b_k)$ is a given k-tuple. The asymptotic formula obtained in this case includes, in particular, Landau's formula for the distribution of square-free numbers with exactly k prime factors [*Handbuch der Lehre von der Verteilung der Primzahlen*, B. G. Teubner, Leipzig, 1909; vol. I, § 56]. Several variants and generalizations of these problems are also discussed.

A theorem of a somewhat different type proved in the last section of the paper states that if $\mathfrak{T}$ is a set of primes with a positive ($x/\log x$)-density, and if the primes of $\mathfrak{T}$ are distributed uniformly among the residue classes (mod n) and prime to n, then this latter property is inherited by the numbers of the form $p_1\cdots p_k$, where k is fixed and $p_1,\cdots,p_k\in\mathfrak{T}$.

The methods used are in part classical and in part constitute a further development of ideas introduced by the author in an earlier publication [Monatsh. Math. **60** (1956), 96–109; MR **18**, 18].

L. Mirsky (Sheffield)

5616:

Subba Rao, M. V. On representation of numbers as sum of two squares. Math. Student **26** (1958), 161–163.

The author proves the classical formula $r_2(n)=4\{d_1(n)-d_3(n)\}$, where $r_2(n)$ is the number of representations of n as the sum of two squares, and $d_1(n)$, $d_3(n)$ are the sums of the divisors of n of the forms $4x+1$, $4x+3$ respectively, by observing that $r_2(n)/4$ and $d_1(n)-d_3(n)$ are both multiplicative functions of n which agree when n is a prime power. The crucial point, that primes of the form $4x+1$ have essentially one representation as the sum of two squares while primes of the form $4x+3$ have no representations as the sum of two squares, is stated as a "well-known result", and the proof depends on this.

M. Newman (Washington, D.C.)

5617:

Zeckendorf, E. Familles de nombres premiers. Bull. Soc. Roy. Sci. Liège **29** (1960), 62–73.

This paper treats the prime numbers $P\equiv\pm1\pmod{12}$, which may be expressed as a difference $|3a^2-b^2|$ of terms relatively prime, and whose products are also of the above form when no prime factor $P'\equiv\pm5\pmod{12}$ is included. The integral values of a and b are obtained by solving equations of the type $b^2-3a^2=k$, where a and b are terms of certain recurring series. Tables are given for the numbers $|3a^2-b^2|$, $|b^2-3|$, and $|3a^2-1|$. Only some of the author's papers are given for references [same Bull. **26** (1957), 112–122; **27** (1958), 28–40, 68–73, 128–141; **29** (1960), 15–27; MR **19**, 730; **20** #1660, 2524a, 2524b; **22** #697].

S. Ikehara (Tokyo)

5618:

Sugunamma, M. Eckford Cohen's generalizations of Ramanujan's trigonometrical sum $C(n,r)$. Duke Math. J. **27** (1960), 323–330.

E. Cohen [same J. **16** (1949), 85–90; MR **10**, 354] defined the generalized Ramanujan sum

$$C^{(s)}(n,r)=\sum_x \exp(2\pi inx/r^s),$$

where x runs through an sth reduced residue system (mod r^s), that is, through all the integers in a complete residue system (mod r^s) whose g.c.d. with r^s has no sth power factor >1. Elsewhere [Amer. Math. Monthly **66** (1959), 105–117; MR **20** #5159] he gave another generalization, namely

$$C_k(n,r)=\sum_{x_1,\cdots,x_k}\exp(2\pi in(x_1+\cdots+x_k)/r),$$

where the summation is extended over all x_j (mod r) such that $(x_1,\cdots,x_k,r)=1$.

In the present paper the author defines

$$C_k^{(s)}(n,r)=\sum_{x_1,\cdots,x_k}\exp(2\pi in(x_1+\cdots+x_k)/r^s),$$

where the summation is over all x_j (mod r^s) such that $(x_1,\cdots,x_k,r^s)$ has no sth power factor >1. It is shown that

$$C_k^{(s)}(n,r)=\sum_{d|g}\mu(r/d)d^{ks},$$

where $g^s=(n,r^s)_s$, the greatest sth power common divisor of n and r^s; moreover

$$C_k^{(s)}(n,r)=\phi_k^{(s)}(r)\mu(r/g)/\phi_k^{(s)}(r/g),$$

where $g^s=(n,r^s)_s$ and $\phi_k^{(s)}(r)$ is the number of sets $x_1,\cdots,$

$x_k \pmod{r^s}$ such that $(x_1, \cdots, x_k r^s)_s = 1$. It follows that $C_s^{(k)}(n, r)$ is multiplicative in r and in r and n, that is,

$$C_k^{(s)}(nn', rr') = C_k^{(s)}(n, r) C_k^{(s)}(n', r') \quad ((nr, n'r') = 1);$$

moreover

$$C_k^{(s)}(n, r) C_k^{(s)}(n', r) = C_{ks}^{(s)}(nn', r)\mu(r).$$

In the final section of the paper the writer evaluates the number of partitions of $n \pmod{r^s}$ of certain types.

L. Carlitz (Durham, N.C.)

5619:

Moser, Leo. Notes on number theory. II. On a theorem of van der Waerden. Canad. Math. Bull. **3** (1960), 23–25.

[For part I see same Bull. **2** (1959), 119–121; MR **21** #4127.] Let $W = W(k, t)$ denote the least integer m such that in every distribution of $1, 2, \cdots, m$ into k classes, at least one class contains an arithmetic progression of $t+1$ terms. By exhibiting an explicit distribution, the present paper proves

$$W(k, t) > tk^{c \log k},$$

where c is a fixed constant. For k sufficiently large compared to t, this is stronger than the result $W(k, t) > (2tk^t)^{1/2}$ of Erdős and Rado [Proc. London Math. Soc. (3) **2** (1952), 417–439; MR **16**, 455].

S. W. Golomb (La Canada, Calif.)

5620:

Schätzer, W. On non-equidistributed sequences of numbers mod. 1. Bol. Soc. Mat. São Paulo **12** (1957), 1–9 (1960).

An unnecessarily complicated proof of the existence of a sequence of numbers in [0, 1] having a particular non-uniform limiting distribution. The desired distribution (constant densities on three subintervals together comprising [0, 1]) is erroneously described, as the integral of the density function is not 1.

W. J. LeVeque (Ann Arbor, Mich.)

5621:

Jenner, W. E. On arithmetical properties of Riemann matrices. Monatsh. Math. **64** (1960), 110–118.

A few remarks (rather simpler in substance than might appear from the text) are made on the discriminant of the order of complex multiplications of abelian function-fields of some special types. *A. Weil* (Princeton, N.J.)

5622:

Negoescu, N. Nombres et points critiques pour les ensembles N_{n+} et M_{n+} dans le problème des approximations asymétriques. I. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **10** (1959), 1–12. (Romanian. Russian and French summaries)

Here is introduced the notion of a critical number for the ensemble of numbers N_{n+} that have a development in a simple infinite continued fraction of the form $\theta = [a_0, a_1, a_2, \cdots]$ with $a_i \geqq n$ $(i \geqq i_0)$, where θ is an irrational number. The following theorem is proved: A necessary condition in order that a number $\theta \in N_{n+}$ be a critical number is that the development of θ in a continued fraction be of the form $[a_0, a_1, a_2, \cdots, a_r, n, b_1, n, b_2, n, b_3, \cdots]$, where the sequence $\{b_i\}$ satisfies one of the three following conditions: (i) $b_i \to \lambda \geqq n$; (ii) $b_i \to +\infty$; (iii) $\liminf b_i = \lambda$ and $\limsup b_i = \lambda + 1$, with λ an integer $\geqq n$.

E. Frank (Chicago, Ill.)

5623a:

Fel'dman, N. I. The measure of transcendency of the number π. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 357–368. (Russian)

5623b:

Fel'dman, N. I. Approximation by algebraic numbers to logarithms of algebraic numbers. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 475–492. (Russian)

In these two important papers, which are closely connected, measures of transcendency for π and $\log \alpha$ (α algebraic) are found which are far better than any obtained before. In the case of π it is shown that there exists an absolute constant Λ_0 such that

$$|\pi - \xi| > H^{-\Lambda_0 n \log (n+2)}$$

for all algebraic numbers ξ, of degree n and height H, provided that $H > \exp(n^2 \log^4(n+2))$. Next let $\alpha_1, \cdots, \alpha_m$ be finitely many algebraic numbers distinct from 0 and such that $\log \alpha_1, \cdots, \log \alpha_m$ are linearly independent over the rational field R. Let $\xi_1, \cdots, \xi_m$ be arbitrary algebraic numbers, of degrees $n_1, \cdots, n_m$ and heights $h_1, \cdots, h_m$ respectively; let n be the degree of the extension field $R(\alpha_1, \cdots, \alpha_m, \xi_1, \cdots, \xi_m)$, and let

$$H = \exp\left\{n\left(\frac{\log h_1}{n_1} + \cdots + \frac{\log h_m}{n_m}\right)\right\}.$$

Then a constant Λ depending only on $\alpha_1, \cdots, \alpha_m$ exists such that

$$\sum_{\mu=1}^{m} |\log \alpha_\mu - \xi_\mu| > H^{-\Lambda (n \log (n+2))^{(m+1)/m}},$$

provided $H > \exp(n^4)$.

The proofs in both papers use similar ideas. They are based on Dirichlet's Schubfachprinzip and on two interpolation formulae that express the coefficients C_{kl} in

$$f(z) = \sum_{k=0}^{q_0-1} \sum_{l=0}^{q-1} C_{kl} z^k e^{lz}$$

either in terms of the values $f(2\pi xi/q)$, where $x = 0, 1, \cdots, qq_0 - 1$, or in terms of the values $f^{(s)}(2\pi xi)$, where $x = 0, 1, \cdots, q_0 - 1$; $s = 0, 1, \cdots, q-1$. Both proofs depend on the fact that for all integers k and l, and for large s, the coefficients $B_{kl}^{(s\kappa)}$ in

$$\left(e^z \frac{d}{dz}\right)^s z^k e^{lz} = e^{(l+s)z} \sum_{\kappa=0}^{k} B_{kl}^{(s\kappa)} z^{k-\kappa}$$

have a greatest common divisor which is likewise large and for which a lower bound can be obtained by means of the prime number theorem. *K. Mahler* (Manchester)

5624:

Ehrhart, Eugène. Sur les polygones réticulaires. C. R. Acad. Sci. Paris **250** (1960), 2986–2988.

If P is a polygon whose vertices are $1/m$ integral points then the number of lattice points in $(km+r)P$ for fixed integer r is a quadratic function of the integer k, the coefficients of k^2 and k being determined by the area and

perimeter of the polygon. Numerical examples. Cf. the preceding series of notes culminating in C. R. Acad. Sci. Paris **250** (1960), 1428–1430 [MR **22** #1563].

J. W. S. Cassels (Cambridge, England)

5625:

Ehrhart, Eugène. Polygones et polyèdres réticulaires. C. R. Acad. Sci. Paris **250** (1960), 3934–3936.

Continuation of #5624; further theorems, numerical examples and conjectures about the number of points with integral coordinates in polyhedra.

J. W. S. Cassels (Cambridge, England)

5626:

Groemer, Helmut. Über lineare homogene diophantische Approximationen. Arch. Math. **11** (1960), 188–191.

Let $F(P)$ denote a distance-function defined for all points $P=(x_1, x_2, \cdots, x_n)$ of the Euclidean space R_n, i.e., $F(P)>0$ for $P\neq(0, 0, \cdots, 0)$, $F(tP)=|t|F(P)$ for any real number t, $F(P+Q)\leq F(P)+F(Q)$. Let $G(P)$ denote a distance-function in R_m. Furthermore let $L_i(x_j)=\sum_{j=1}^m a_{ij}x_j$ $(i=1, 2, \cdots, n)$ denote n linear forms with real coefficients. Then the author proves the following theorem.

If F, G and L_i $(i=1, 2, \cdots, n)$ are given, there exists an infinity of systems of integers x_j $(j=1, 2, \cdots, m)$, and for every system $\{x_j\}$ integers y_i $(i=1, 2, \cdots, n)$, such that

$$F^n(L_i(x_j)-y_i)G^m(x_j) < \left(\frac{n}{n+m}\right)^n\left(\frac{m}{n+m}\right)^m\binom{n+m}{m}\frac{2^{n+m}}{V(F)V(G)},$$

where $V(F)$ and $V(G)$ denote the volumes of the convex bodies given by $F\leq 1$ and $G\leq 1$.

The author gives a short direct proof using Minkowski's lattice theorem. He observes that this problem is connected with a more general theory developed by Mullender [Nederl. Akad. Wetensch. Proc. **51** (1948), 874–884; MR **10**, 285]. In fact it is possible to give also a simple proof using Mullender's ideas.

Finally, the author gives some applications.

Corrections: The right-hand member of the inequality in Satz 3 (p. 191) is erroneous: The factors

$$\left(\frac{n}{n+m}\right)^n\left(\frac{m}{n+m}\right)^m\binom{n+m}{m}\left(\frac{2}{\pi}\right)^{n+m}$$

should be replaced by

$$\left(\frac{n}{n+m}\right)^{2n}\left(\frac{m}{n+m}\right)^{2m}\binom{2n+2m}{2m}\left(\frac{4}{\pi}\right)^{n+m}.$$

Moreover, in the next inequality, due to Minkowski, the right-hand member should read

$$\left(\frac{n}{n+1}\right)^{2n}\frac{2n+1}{n+1}\left(\frac{4}{\pi}\right)^{n+1}.$$

J. Popken (Amsterdam)

5627:

Schmidt, Wolfgang. Masstheorie in der Geometrie der Zahlen. Acta Math. **102** (1959), 159–224.

Let S be a Borel set in R^n, $\Delta(S)$ its critical determinant, $V(S)$ its volume. By integrating with respect to the measure induced in the space of lattices by the Haar measure, Siegel [Ann. of Math. (2) **46** (1945), 340–347; MR **6**, 257] proved a mean value theorem about the average number of lattice points in S, from which Hlawka's theorem that $V(S)/\Delta(S)\geq 1$ follows immediately as a corollary.

A natural method of proceeding further would seem to be to obtain estimates for the number of k-tuples of lattice points in K. For $k\leq n-1$, methods similar to Siegel's, using integration in the same space, were used by Rogers and the author, from which considerable improvements on Hlawka's result were derived for large n.

Then [Monatsh. Math. **61** (1957), 269–276; **62** (1958), 250–258; MR **20** #1672, 5769] the author obtained a mean value formula for the n-tuples. In this case the integral involves not only the Haar measure, but also integration with respect to some extra parameters. In the present paper, this mean value theorem is generalized to k-tuples, where $k\geq n$. With the aid of the new formula, an expression is found for the Haar measure of the set of admissible lattices. Considerable difficulties of convergence arise in the proof, since the obvious term-by-term integration leads to a divergent series (as the author actually shows for $n=2$). These convergence difficulties are overcome rather ingeniously by ordering the points in R^n. A careful study is also needed of certain sums involving the abelian factor groups modulo the sublattices generated by the k-tuples of lattice points, using the "Möbius function" of Delsarte.

In the latter part of the paper, these results are used to deduce a still further improvement of the Minkowski-Hlawka theorem: $V(S)/\Delta(S)$ is not less than $nr-2$, where $r=0.278\cdots$, for sufficiently large n. Elegant applications in 2-space include an exact value, in the form of a finite series, for the measure of the set $A(S)$ of S-admissible lattices in the case when S is an annulus. A corollary is deduced about the numbers represented by a positive binary quadratic form.

Applications are also made to the problem of lattice coverings of space, using ideas of Rogers, though the author has added a note in proof to the effect that his results in this direction have been superseded by a recent paper of Rogers himself.

Two other results worth mentioning are: (i) If S has infinite volume, then almost all lattices have infinitely many points in S; (ii) The measure of the set $A(S)$ is a continuous function of S in the topology defined by the measure of the symmetric difference of subsets of R^n.

A. M. Macbeath (Dundee)

FIELDS

5628:

Jaffard, Paul. ★Les systèmes d'idéaux. Travaux et Recherches Mathématiques, IV. Dunod, Paris, 1960. ix+132 pp. 25 NF.

This book provides an excellent presentation of the theory of ideal systems in partially ordered abelian groups and its application to the theory of divisibility in fields with respect to an order. The starting points of this theory were the v-ideals of van der Waerden-Artin and a systematic discussion of ideal systems in integral domains by H. Prüfer, and the main contributor was P. Lorenzen who gave in 1939 an axiomatic treatment and a generalization to partially ordered groups. Since then this method has proved to be very fruitful and it is pleasing to have the

whole theory available in this well and clearly written monograph.

Instead of a field F with a prescribed order A in it, one considers the multiplicative group G of F and makes it into a partially ordered group by agreeing to put $a\leqq b$ whenever $ba^{-1}\in A$. Then the addition of F is eliminated and the role of the Dedekind ideals may be played by a system of r-ideals defined by a correspondence $X\rightarrow X_r$ (with X ranging over the nonvoid subsets of G bounded from below) such that: (1) $X\subseteq X_r$; (2) $X\subseteq Y_r$ implies $X_r\subseteq Y_r$; (3) $(a)_r$ is the set of all x satisfying $x\geqq a$; (4) $aX_r=(aX)_r$ for all $a\in G$. By specializations one obtains the s-, v-, d- and t-ideal systems which may have properties designated by $\alpha, \beta, \gamma, \delta$; the interrelations between these properties of the systems are investigated. Particular attention is paid to the case in which G is simply or lattice-ordered or is a subdirect union of simply ordered groups. The discussion of these and other themes is based on the facts that to any theorem in G there corresponds one in F and that several problems can be dealt with better in G than in F. This method is adapted even in cases when restriction to Dedekind ideals seems to be necessary, in order to obtain simpler proofs of results which have not formerly been incorporated into this theory.

The following chapter headings, each followed by the contents of the sections, will give an idea of what is covered: (I) Généralités sur les groupes ordonnés et les systèmes d'idéaux (Groupes ordonnés, Homomorphismes croissants, Systèmes d'idéaux, Propriétés des systèmes d'idéaux); (II) Théorie des réalisations (Réalisations des groupes réticulés, Les groupes réticulés associés à un groupe ordonné, Empreintes d'un système d'idéaux); (III) Groupes et anneaux particuliers (Les groupes totalement ordonnés et les anneaux de valuation, Groupes et anneaux normaux); (IV) Les théorèmes de permanence (Anneaux de fractions et anneaux quotients, Anneaux de polynômes, Extensions algébriques); (Appendice) Les x-idéaux.

This book will be of great value for any one dealing with this topic. *L. Fuchs* (Budapest)

5629:

Ennola, Veikko. Two elementary proofs concerning simple quadratic fields. Nordisk Mat. Tidskr. **6** (1958), 114–117, 136.

Es werden zwei neue Beweise, ein etwas mühsamer elementarer und ein auf dem Dirichletschen Satz fußender sehr kurzer Beweis des Satzes angegeben, daß im durch $\sqrt{m}$ erzeugten Zahlkörper ($m\neq\pm 1$ ganze rationale Zahl ohne mehrfache Primteiler) nur dann die Primzerlegung der Elemente möglich ist, wenn $|m|$ eine Primzahl oder das Produkt von zwei Primzahlen ($\not\equiv 1 \pmod 4$) ist. Frühere Beweise liegen bei Behrbohm und Rédei [J. Reine Angew. Math. **174** (1936), 192–205] und Inkeri [Ann. Univ. Turku. Ser. A **9** (1948), no. 1; MR **10**, 236] vor. Ref. bemerkt, daß der schärfere, elegante Satz gilt, wofür er einen kurzen elementaren Beweis mitteilen wird [siehe folgende Bespr.], der aus dem ersten Beweis des Verf. entstand: Die Diskriminante eines (absolut) quadratischen Zahlkörpers mit Primzerlegung ist entweder eine Stammdiskriminante oder das Produkt von zwei negativen Stammdiskriminanten. (Stammdiskriminanten heißen -4, ± 8, ferner die positiven und negativen Primzahlen p mit $4|p-1$.) Des Verf. erster Beweis gilt auch für den schärferen Satz. *L. Rédei* (Zbl **84**, 39)

5630:

Rédei, Ladislaus. Über die quadratischen Zahlkörper mit Primzerlegung. Acta Sci. Math. Szeged **21** (1960), 1–3.

The author calls the integers -4, ± 8, $\pm p$ ($\equiv 1 \pmod 4$), p a rational prime) primitive discriminants (Stammdiskriminanten). The discriminant D of a quadratic field (over the rationals) is uniquely the product of primitive discriminants. The author gives a proof of the following known theorem: The discriminant D of a quadratic field with class number 1 is either a primitive discriminant or the product of two negative primitive discriminants. This combines into a single statement two known theorems concerning real and imaginary quadratic fields with unique factorization.

The very simple proof given is an adaptation of one given by Veikko Ennola [see preceding review] and is based on the following simple lemma: If Q denotes the ring of integers of the quadratic field and if $p|D$, then there exists a unit η such that $\sqrt{(p\eta)}\in Q$.

R. Ayoub (University Park, Pa.)

5631:

Rotman, Joseph J. A note on completions of modules. Proc. Amer. Math. Soc. **11** (1960), 356–360.

Modules M are considered over a discrete valuation ring R such that $\bigcap_n p^nM=0$. M is a topological module by taking p^nM as a system of neighbourhoods of 0. Theorems: Complete modules are completions of their basic submodules (basic in the sense of Kulikov). If M is torsion-free of finite rank with basic submodule B, then $\operatorname{rank}_{R^*} M^*=\operatorname{rank}_R B$ (asterisks indicate completions). A pure complete submodule is always a direct summand. If S is a closed submodule of a complete M, then M/S is again complete. Any homomorphism of a basic submodule of a complete M into a complete module can be uniquely extended over M. Of these the known results are given with new simple proofs. *L. Fuchs* (Budapest)

5632:

Rotman, Joseph. Mixed modules over valuation rings. Pacific J. Math. **10** (1960), 607–623.

The Kaplansky-Mackey generalization [I. Kaplansky and G. W. Mackey, Summa Brasil. Math. **2** (1951), 195–202; MR **14**, 128] of Ulm's theorem is further generalized in various ways by analyzing the proof. A module M over a discrete valuation ring is said to have the coset property if, for each finitely generated submodule S of M, the coset $x+S$ contains an element of maximal height. It is shown that two reduced countably generated modules M and M' of finite (torsion-free) rank r and with the coset property are isomorphic if and only if they have the same Ulm invariants and an additional condition is satisfied; this may be replaced by the requirement that torsion modules T and T' exist such that $T\oplus M\cong T'\oplus M'$. This structure theorem is applied in the case $r=1$ to solve the square root problem and in another case to prove a cancellation law. The final section deals with completely decomposable modules. *L. Fuchs* (Budapest)

ABSTRACT ALGEBRAIC GEOMETRY

5633:

Abhyankar, Shreeram. Über die endliche Erzeugung der Fundamentalgruppe einer komplex-algebraischen Mannigfaltigkeit. Math. Ann. **139**, 265–274 (1960).

The author proves the following theorem without the use of a triangulation: Let V be an irreducible projective algebraic variety. If then W is a proper algebraic subvariety of V, such that $V-W$ is analytically irreducible at each point, then $\pi_1(V-W)$ is finitely generated (π_1 denoting the fundamental group).

Previous proofs of this quite general theorem all use the triangulation theorem for varieties, which is, to say the least, a very complicated theorem. The proof presented here is in part inspired by the work of Grauert and Remmert. *R. Bott* (Cambridge, Mass.)

5634:

Chevalley, C. La théorie des groupes algébriques. Proc. Internat. Congress Math. 1958, pp. 53–68. Cambridge Univ. Press, New York, 1960.

An expository paper, covering the theory of abelian varieties, and the theory of algebraic groups of matrices (called linear groups), over a field of arbitrary characteristic. The main structure theorem, stating that any algebraic group (=group-variety) is an extension of a linear group by an abelian variety, is also mentioned. The sections dealing with abelian varieties cover the period 1948–1952 (including the Picard variety), plus the duality theorem (1958) and class field theory over abelian varieties (1956–58). Linear groups and their Lie algebras, including the contributions by the author, are covered up to 1958. No list of references is supplied.

I. Barsotti (Providence, R.I.)

5635:

Ono, Takashi. On some arithmetic properties of linear algebraic groups. Ann. of Math. (2) **70** (1959), 266–290.

In § 1, some lemmas are proved concerning toruses (in the sense of the theory of algebraic groups, cf. A. Borel, same Ann. **64** (1956), 20–82 [MR **19**, 1195]). Some essential concepts are then introduced concerning the arithmetic theory of an algebraic group G, defined over the rational number-field **Q**, and isomorphic to a subgroup of a linear group. Let G_A be the adèle-group for G (here called the group of G-idèles, and denoted by $J(G)$; for the definition, cf. T. Ono, Bull. Soc. Math. France **85** (1957), 307–323 [MR **20** #880]). Let $G_A{}^1$ be the subgroup of G_A consisting of the elements $x \in G_A$ such that $|\chi(x)|=1$ for every rational character χ of G (i.e., every rational representation, defined over **Q**, of G into the multiplicative group G_m in one variable; $|\ |$ is the idèle-module). Let $G_{\mathbf{Q}}$ be the group of elements of G with coordinates in **Q**. For a given representation of G as a subgroup of a linear group, let $G_A{}^\infty$ be the subgroup of G_A consisting of the elements of G_A whose p-adic components are all units (i.e., matrices with coefficients in the ring of p-adic integers, with a determinant equal to a p-adic unit). The author says that G is of type (F) if G_A consists of finitely many double cosets $G_{\mathbf{Q}}\backslash G_A/G_A{}^\infty$ with respect to $G_{\mathbf{Q}}$ and $G_A{}^\infty$; of type (C) if $G_A{}^1/G_{\mathbf{Q}}$ is compact; of type (M) if it is of finite measure for the Haar measure on $G_A{}^1$ (this being shown to be both right and left invariant). It is shown that, if G is of type (C), it is of type (F) (obviously, it is of type (M)), and that its group of units, $G_{\mathbf{Q}} \cap G_A{}^\infty$, is finitely generated. If G is a semidirect product of two connected (i.e., irreducible) groups G', G'' (G' normal, $G''=G/G'$), and if G' has no rational characters and is of type (C), then G is of type (C), respectively (F), respectively (M), whenever G'' is so. The main theorems of the paper are then Theorem 3, stating that every connected solvable group is of type (C), and Theorem 5, which gives the rank of the group of units of all such groups (the latter is a wide generalization of the Dirichlet unit-theorem). The paper also includes a few results on non-connected algebraic groups. *A. Weil* (Princeton, N.J.)

5636:

Snapper, Ernst. Polynomials associated with divisors. J. Math. Mech. **9** (1960), 123–139.

Let X be an irreducible, normal, projective variety and let F be an algebraic coherent sheaf on X. Let $D_1, \cdots, D_n$ be divisors on X which are everywhere locally linearly equivalent to zero. Then the Euler-Poincaré characteristic $\chi(X, F(x_1D_1+\cdots+x_nD_n))$ is a polynomial in $x_1, \cdots,$ x_n (rational integers) [E. Snapper, same J. **8** (1959), 967–992; MR **22** #44]. After setting

$$[i_1, \cdots, i_n] = \prod_k \binom{x_k+i_k-1}{i_k} \quad (\text{for } i_k \geq 0),$$

the above polynomial is expanded as a linear combination of the polynomials $[i_1, \cdots, i_n]$. The virtual arithmetic genus of a virtual subvariety [cf. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie*, Springer, Berlin, 1956; MR **18**, 509; p. 133] is the coefficient of $[1, \cdots, 1]$ in the above expansion.

M. Nagata (Kyoto)

5637:

Igusa, Jun-ichi. Arithmetic variety of moduli for genus two. Ann. of Math. (2) **72** (1960), 612–649.

This paper presents a first new consequence of a novel way of looking at the moduli of an algebraic variety: for each characteristic p (zero or a prime), let Ω_p be a "universal domain" (=algebraically closed field of infinite absolute transcendency) of characteristic p, and let Ω be the union of the Ω_p; let $\mathscr{C}$ be (in this case) the "set" of all nonsingular curves of genus 2 defined over the various Ω_p. An "absolute invariant" is a mapping φ of $\mathscr{C}$ into Ω such that: (a) $\varphi C \in \Omega_p$ if C is defined over Ω_p; (b) $\varphi C = \varphi C'$ if and only if C and C' are birationally equivalent; (c) if C' is a specialization of C (possibly of unequal characteristic), $\varphi C'$ is a specialization of φC. The set of absolute invariants forms, in an obvious way, a ring $\mathfrak{R}$, and the main object of the paper is the study of the structure of $\mathfrak{R}$. At the present state of the art, the only way of establishing this structure seems to consist in finding suitable normal forms for the varieties concerned (curves of genus 2), and then extracting birational invariants. If $p \neq 2$, the birational equivalence of two curves in normal form is translated, as in the classical case, into the projective equivalence of two divisors of order 6 on a straight line (divisors consisting of the projections of the 6 Weierstrass points); and this in turn leads to the theory of invariants of a binary sextic form; it so happens that these invariants work also when $p=2$.

The main results are: $\mathfrak{R}$ is a finitely generated (over the ring N of integers) integral domain; if k is a subfield of Ω_p, $\mathfrak{R}\otimes_N k$ is integrally closed; hence the set $\mathfrak{M}_k$ of the homomorphisms of $\mathfrak{R}\otimes_N k$ into Ω_p is an affine normal variety; this variety is rational, has dimension 3, and has exactly 1 singular point if $p\neq 2$, or one rational singular curve if $p=2$.

The paper closes with some special results on curves "with many automorphisms". The reader should be warned that the proof of lemma 8 contains one statement (the irreducibility of X') and one tacit assumption (the irreducibility of L') whose proofs are not at all obvious.

I. Barsotti (Providence, R.I.)

LINEAR ALGEBRA

See also 5579, B6072.

5638:

Froda, Alexandru. Réduction des formes quadratiques. Acad. R. P. Romîne. Stud. Cerc. Mat. **11** (1960), 49–59. (Romanian. Russian and French summaries)

This note describes how to reduce a real symmetric quadratic form to a sum of real multiples of squares by associating to the given form in a natural way a hermitian form and using the reduction theory for hermitian forms.

M. M. Day (Urbana, Ill.)

5639:

Mirsky, L. Symmetric gauge functions and unitarily invariant norms. Quart. J. Math. Oxford Ser. (2) **11** (1960), 50–59.

Let $\mathfrak{M}_n$ denote the linear space of all matrices of order n with complex elements. According to J. von Neumann [Mitt. Forsch.-Inst. Math. Mech. Univ. Tomsk **1** (1937), 286–299], a norm $\|\cdot\|$ on $\mathfrak{M}_n$ is said to be unitarily invariant, if $\|XU\|=\|UX\|=\|X\|$ for all $X\in\mathfrak{M}_n$ and all unitary matrices U of order n. Let A, $B\in\mathfrak{M}_n$. Let $\alpha_1\geqq\cdots\geqq\alpha_n$ and $\beta_1\geqq\cdots\geqq\beta_n$ be the singular values of A, B respectively. Let $C=\operatorname{diag}(\alpha_1-\beta_1,\cdots,\alpha_n-\beta_n)$ and $D_k=\operatorname{diag}(0,\cdots,0,\alpha_{k+1},\cdots,\alpha_n)$. For any unitarily invariant norm $\|\cdot\|$ on $\mathfrak{M}_n$, it is shown that: (1) $\|A-B\|\geqq\|C\|$; (2) $\|D_k\|$ is the infimum of $\|A-X\|$, when X runs over all matrices in $\mathfrak{M}_n$ with rank $\leqq k$ (or with rank $=k$).

Ky Fan (Detroit, Mich.)

5640:

Robinson, D. W. A note on matrix commutators. Michigan Math. J. **7** (1960), 31–33.

Let D be a nonsingular n-by-n matrix over a field F of characteristic zero or prime $p>n$. It is shown that there exist nonsingular matrices A and B over F for which $D=ABA^{-1}B^{-1}$ and $A(AB-BA)=(AB-BA)A$ if and only if either (i) $D-I$ is nilpotent or, in case F has more than two elements, (ii) $D-I$ is similar to $I-D^{-1}$.

C. R. Putnam (Lafayette, Ind.)

5641:

Marcus, Marvin; Westwick, Roy. Linear maps on skew symmetric matrices: the invariance of elementary symmetric functions. Pacific J. Math. **10** (1960), 917–924.

Let S_n be the space of $n\times n$ skew symmetric real matrices. If $A\in S_n$, let $E_{2k}(A)$ denote the sum of the $2k$-rowed principal minors of A, that is, the $2k$th elementary symmetric function of the latent roots of A. If U is a fixed real orthogonal matrix, the mapping $A\to UAU'$ is a linear transformation of S_n onto itself. As is well-known, it has the property that $E_{2k}(UAU')=E_{2k}(A)$ for all k. The authors propose to determine those linear transformations of S_n into itself for which the equation $E_{2k}(S(A))=E_{2k}(A)$ holds for all A, but only for one particular value of k. Their results are as follows. (i) If $4\leqq 2k\leqq n$ and $n\geqq 5$, then there exists a real matrix P and a real number α such that $S(A)=\alpha P'AP$, where $\alpha PP'=I$ if $2k<n$ and $\alpha PP'$ is unimodular if $2k=n$. (ii) If $2k=n=4$, there is an additional transformation which is given explicitly in the paper.

W. Ledermann (Manchester)

5642:

Cherubino, Salvatore. Sulle radici r^{me} e sul quasi-logaritmo di una matrice singolare. Ann. Scuola Norm. Sup. Pisa (3) **13** (1959), 373–388.

The first part of this paper proves a number of related results concerning the rth roots of a square matrix, using the author's modification of the Jordan canonical form.

The second part, which is essentially unrelated to the first, is an attempt to extend to singular matrices the author's previous work on the logarithm of a matrix. The quasi-logarithm of a singular matrix is defined to be the logarithm of its nucleus, the nucleus of a matrix being the modified matrix obtained by replacing any zeros by units in the leading diagonal of the Jordan canonical form. The author unfortunately does not demonstrate the possible utility of the quasi-logarithm apart from the rather obvious fact that it coincides with the logarithm in the case of a non-singular matrix.

D. E. Rutherford (St. Andrews)

5643:

Jongmans, F. Problèmes matriciels liés au rang. Bull. Soc. Roy. Sci. Liège **29** (1960), 51–60.

This paper uses elementary transformations of matrices and simple inequalities concerning rank to discuss certain linear matric equations. The principal results are as follows. Let A, B, C be matrices of type $l\times m$, $n\times p$, $l\times p$ and rank a, b, c respectively. (i) The equation $AXB=C$ is soluble if and only if the rank of (A, C) is equal to a and the rank of $\begin{pmatrix}B\\C\end{pmatrix}$ is equal to b. (ii) The linear variety of solutions is of dimension $mn-ab$. (iii) The rank x of any solution X satisfies the inequality

$$(*)\qquad c\leq x\leq \min(m, n, m+n+c-a-b).$$

(iv) If x is any integer which satisfies (*) and if the equation $AXB=C$ is soluble, then it possesses a solution of rank x.

{Reviewer's remark: The proof of (iv) is needlessly complicated, since the assertion follows almost immediately from results concerning the more special equations $AX=C$ and $XB=C$ treated in earlier sections of the paper.}

L. Mirsky (Sheffield)

5644:

Muir, Thomas. ★A treatise on the theory of determinants. Revised and enlarged by William H. Metzler. Dover Publications, Inc., New York, 1960. vii+766 pp. $2.95.

Paperback re-issue of the Longmans, Green edition of 1928.

ASSOCIATIVE RINGS AND ALGEBRAS

See also 5631, 5632.

5645:

Kustaanheimo, Paul. On a tensor ring notation combining the vector and Cracovian notations. Ann. Acad. Sci. Fenn. Ser. A I No. 262 (1959), 6 pp.

5646:

Herstein, I. N.; Kleinfeld, Erwin. Lie mappings in characteristic 2. Pacific J. Math. **10** (1960), 843–852.

Let φ be an additive mapping from a simple ring R onto a simple ring $R' \neq 0$ of characteristic 2 which satisfies either (A) $\varphi(x^2)=\varphi(x)^2$ and $\varphi(x^3)=\varphi(x)^3$ for all $x \in R$, or (B) $\varphi(xy-yx)=\varphi(x)\varphi(y)-\varphi(y)\varphi(x)$ and $\varphi(x^3)=\varphi(x)^3$ for all $x, y \in R$. The authors prove that with either hypothesis (A) or (B), φ is either an isomorphism or an anti-isomorphism. This result settles a question left unsolved in previous papers by Herstein [Trans. Amer. Math. Soc. **81** (1956), 331–341; MR **17**, 938] and Smiley [ibid. **84** (1957), 426–429; MR **18**, 715]. The proof is by arguments similar to those in the previous papers, but in this case Herstein's results on the Lie and Jordan rings of a simple associative ring [Amer. J. Math. **77** (1955), 279–285; MR **16**, 789] are also needed. *C. W. Curtis* (Madison, Wis.)

5647:

Kertész, A.; Steinfeld, O. On the symmetry of semi-simple rings. Amer. Math. Monthly **67** (1960), 450–452.

A direct proof that a ring with a right unit which has a decomposition into a direct sum of a finite number of minimal left ideals has a left unit and a decomposition into a direct sum of a finite number of minimal right ideals. *Graham Higman* (Chicago, Ill.)

5648:

Gerstenhaber, Murray; Yang, C. T. Division rings containing a real closed field. Duke Math. J. **27** (1960), 461–465.

The authors prove the following theorem: If D is a division ring containing a real closed subfield R such that D has finite left dimensionality $[D:R]_L$ over R, then either $D=R$, $D=R(i)$, $i^2=-1$, or D is a quaternion algebra over a real closed field F such that $F(i)=R(i)$. {A simpler proof of this result can be obtained by using the following theorem of the reviewer's: If D is a division ring which contains a subfield R such that $[D:R]_L<\infty$, then $[D:F]<\infty$ for F the center [Amer. J. Math. **77** (1955), 593–599; MR **17**, 9]. Thus if R is real closed let K be the subfield generated by R and F. We have $[K:R]<\infty$ and $[K:F]<\infty$. Then $K(i)$ is algebraically closed and F is a subfield such that $[K(i):F]<\infty$. Hence, by a theorem of Artin's, F is either algebraically closed or is real closed. The result now follows from Frobenius' theorem.} *N. Jacobson* (New Haven, Conn.)

5649:

Faith, Carl C. On conjugates in division rings. Canad. J. Math. **10** (1958), 374–380.

Es sei D ein von seinen Zentrum C verschiedener Schiefkörper, A ein echter Unterschiefkörper von D mit $A \not\subseteq C$, D^* die multiplikative Gruppe der von Null verschiedenen Elemente aus D, und $H(A)$ die Untergruppe der Elemente $t \in D^*$ mit $tAt^{-1}=A$. Dann wird gezeigt, daß $H(A)$ keinen endlichen Index in D^* besitzt. Damit äquivalent ist die Aussage, daß es unendlich viele verschiedene konjugierte Unterschiefkörper xAx^{-1} mit $x \in D^*$ gibt. Der Beweis beruht auf dem Wedderburnschen Satz über die Kommutativität der endlichen Schiefkörper und einem Satz von N. Jacobson über algebraische Schiefkörpererweiterungen eines endlichen Körpers. Das Resultat wird noch dadurch ergänzt, daß unter gewissen Voraussetzungen die Existenz von unendlich vielen Schiefkörpern xAx^{-1} für die Elemente x aus einer Teilmenge von D^* nachgewiesen werden kann. *F. Kasch* (Zbl **82**, 32)

5650:

Jenner, W. E. On non-associative algebras associated with bilinear forms. Pacific J. Math. **10** (1960), 573–575.

Let $u_1, \cdots, u_n$ span the vector space V_0 over the (commutative) field k. Let $f_0(\ ,\ )$ be a bilinear form over V_0. The algebra A with basis $e_0, e_1, \cdots, e_n$ over k is defined as follows: e_0 is a unit; $e_ie_j=f_0(u_i, u_j)e_0$, $i, j>0$. If the bilinear form f given by $f(e_i, e_j)=f_0(u_i, u_j)$ is nondegenerate, and if $n>1$, then: (α) A has no proper left or right ideals; (β) The group of automorphisms of A is isomorphic to the group G of those linear transformations of V which leave f invariant; when $k=R$ (the field of reals), the Lie algebra of G is $\mathscr{D}(A)$ (derivations of A); (γ) Suppose f to be degenerate, and $f(v, w)=f(w, v)$; then A is the sum of a nilpotent ideal and of a semisimple subalgebra. *J. L. Brenner* (Madison, Wis.)

5651:

Zykov, A. A. Algebras of complexes. Amer. Math. Soc. Transl. (2) **15** (1960), 15–32.

Translation of Mat. Sb. (N.S.) **41** (**83**) (1957), 159–176 [MR **19**, 246].

5652:

Nöbauer, Wilfried. Zur Theorie der Vollideale. Monatsh. Math. **64** (1960), 176–183.

The present paper continues the study of full ideals (Vollideale) initiated in two earlier papers [Math. Ann. **134** (1958), 248–259; J. Reine Angew. Math. **201** (1959), 207–220; MR **20** #4549; **21** #7225]. For definitions and notation assumed in this review see in particular the review of the second paper cited; we note that the ideal $\mathfrak{a}$ corresponding to a given full ideal A is called the enveloping ideal [Umschliessungsideal). The following results are obtained. (1) The sum and intersection of full ideals of R are again full ideals, and the enveloping ideals are obtained from the enveloping ideals of the given full ideals by taking sums or intersections, respectively. (2) Let V be a full ideal with enveloping ideal $\mathfrak{v}$ and let $\mathfrak{v}=\mathfrak{a}\mathfrak{b}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are relatively prime. Then $V=AB$, where $\mathfrak{a}$ is the enveloping ideal of A and $\mathfrak{b}$ is the enveloping ideal of B. (3) Let ϑ denote a homomorphism of the ring $\mathfrak{r}$ onto the ring $\mathfrak{t}$; then ϑ induces a homomorphism Θ of R onto $T=\mathfrak{t}(\mathfrak{x})$ by means of

$$\Theta \sum a_{r_1\cdots r_k}x_1^{r_1}\cdots x_k^{r_k} = \sum (\vartheta a_{r_1\cdots r_k})x_1^{r_1}\cdots x_k^{r_k}.$$

It is proved that the image of the full ideal V under Θ is a full ideal W of T, and that if $\mathfrak{v}$ is the enveloping ideal of V then $\vartheta\mathfrak{v}$ is the enveloping ideal of W. (4) A converse of (3). (5) Let $\mathfrak{t}=\mathfrak{r}/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal of $\mathfrak{r}$. If V is a full

ideal of R with enveloping ideal $\mathfrak{a}$, then ΘV is a full ideal of T with enveloping ideal (0). Put $\Lambda(\Theta V)=V$. It is proved that one obtains all full ideals of R with enveloping ideal $\mathfrak{a}$ by taking ΛW, where W runs through the full ideals of T with enveloping ideal (0).

Finally, the author shows how from a given full ideal V additional full ideals can be obtained. Indeed let V' denote the set of polynomials $f(x)\in R$ such that $f(x)\in V$, $\partial f/\partial x_s\in V$ $(s=1,\cdots,k)$; then V' is a full ideal of R which is called the derivative of V. More generally, define $V^{(n)}$ recursively by means of $V^{(n)}=(V^{(n-1)})'$. Then $V^{(n)}$ is the set of all polynomials $f(x)\in V$ such that all partial derivatives of order $\leqq n$ are contained in V. It is proved that all the derivatives $V^{(n)}$ have the same enveloping ideal as V. Also the intersection $D=V\cap V'\cap V''\cap\cdots$ is a full ideal with the same enveloping ideal as V; and D coincides with all its derivatives. If V and W are full ideals, then $(V\cap W)'=V'\cap W'$. For all $n\geqq 1$ we have $V^{(n)}\subseteq V^{(n-1)}$, where $V^{(0)}=V$.

The paper closes with a discussion of the example $V=((x^{p^s}-x)^m, p)$ over the ring of integers.

L. Carlitz (Durham, N.C.)

5653:

Northcott, D. G. Semi-regular rings and semi-regular ideals. Quart. J. Math. Oxford Ser. (2) **11** (1960), 81–104.

Soient A un anneau noethérien et $\mathfrak{a}$ un idéal $(\neq A)$ de A; appelons hauteur de $\mathfrak{a}$, et notons $h(\mathfrak{a})$, la plus petite des hauteurs des idéaux premiers associés de $\mathfrak{a}$. Si $\mathfrak{a}$ est engendré par n éléments, on a $h(\mathfrak{a})\leqq n$, d'après Krull. On dit que A est un anneau de Macaulay si, pour tout idéal $\mathfrak{a}\neq A$ engendré par $h(\mathfrak{a})$ éléments, les idéaux premiers de $\mathfrak{a}$ sont tous de hauteur $h(\mathfrak{a})$; les $h(\mathfrak{a})$ générateurs de $\mathfrak{a}$ forment alors une suite première. La propriètè d'être un anneau de Macaulay est locale [cf. Northcott et Rees, J. London Math. Soc. **32** (1957), 367–374; MR **19**, 630; ces anneaux y sont appelés "semi-regular"]. Étude des anneaux de fractions et des chaînes d'idéaux premiers des anneaux de Macaulay. La technique d'adjonction d'indéterminées adaptée aux anneaux de Macaulay est la suivante: dans l'anneau de polynomes $R=A[X_1,\cdots,X_n]$ les polynomes $f(X)$ dont les coefficients engendrent l'idéal unité A forment une partie multiplicativement stable S, et on forme R_S. Un idéal $\mathfrak{a}$ de A est dit semi-régulier si $A/\mathfrak{a}$ est un anneau de Macaulay. Si A est un anneau de Macaulay, et si l'idéal $\mathfrak{a}$ peut être engendré par $h(\mathfrak{a})$ éléments, les idéaux $\mathfrak{a}^n$ sont semi-réguliers. Soit enfin (a_{ij}) une matrice à m lignes et p colonnes $(m\leqq p)$ sur un anneau de Macaulay A; notant $\mathfrak{b}$ l'idéal engendré par les déterminants d'ordre m extraits de cette matrice, et supposant $\mathfrak{b}\neq A$, on a $h(\mathfrak{b})\leqq p-m+1$; si, de plus, on a $h(\mathfrak{b})=p-m+1$, alors $\mathfrak{b}$ est semi-régulier. Tout ceci généralise des résultats de Macaulay.

P. Samuel (Clermont-Ferrand)

5654:

Jans, J. P. Verification of a conjecture of Gerstenhaber. Proc. Amer. Math. Soc. **11** (1960), 335–336.

Let T_n be the algebra of all strictly triangular matrices with coefficients in an arbitrary field. The author proves that if M is a faithful $T_n/T_n{}^k$-module $(3\leqq k\leqq n-1)$, then $\dim M\geqq 3+(k-2)(n-k+1)$. In particular, since $\dim M>n$, this proves the conjecture of M. Gerstenhaber [Ann. of Math. (2) **70** (1959), 167–205; MR **22** #4743; p. 168] that $T_n/T_n{}^k$ $(n\geqq 4)$ cannot be represented faithfully by $n\times n$ matrices.

S. A. Amitsur (Jerusalem)

5655:

Skornyakov, L. A. Projective mappings of modules. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 511–520. (Russian)

Let F be an associative ring with unit element and A a unitary left F-module. An element $a\in A$ is called free if $\lambda a=0$ implies $\lambda=0$. A is called admissible if the following two conditions are fulfilled: (A) For any three elements $x,y,z\in A$ there is a free element $w\in A$ with $(Fx+Fy+Fz)\cap Fw=0$. (B) If $x,y,u,t\in A$, and if x,y,u are free elements and $Fx\cap Fy\neq 0$, $Fu\cap Ft\neq 0$, there is a free element $w\in A$ with $Fw\cap Fx=Fw\cap Fy=Fw\cap Fu=Fw\cap Ft=0$. The isomorphism $S\to S^*$ of the lattice of all finitely generated submodules of A onto the lattice of all finitely generated submodules of the unitary G-module B will be called a projectivity of A upon B if for any $a\in A$, $b\in B$ there are $b'\in B$, $a'\in A$ with $(Fa)^*=Gb'$, $(Fa')^*=Gb$, and if, in addition, there exist free elements $u\in A$, $v\in B$ with $(Fu)^*=Gv$. A semi-linear transformation of A upon B is a pair (σ,τ) consisting of an isomorphism σ of the additive group A upon the additive group B and an isomorphism τ of the ring F upon the ring G such that $(\alpha a)\sigma=(\alpha\tau)(a\sigma)$ holds with every $\alpha\in F$, $a\in A$. The following theorem is proved. Let F be an associative ring with unit element 1 for which $\alpha\beta=1$ implies $\gamma\alpha=1$ for a suitable $\gamma\in F$. Then every projectivity of the admissible F-module A upon the G-module B is induced by a semi-linear transformation. This theorem is a considerable generalization of the so-called first fundamental theorem of projective geometry [R. Baer, *Linear algebra and projective geometry*, Academic Press, New York, 1952; MR **14**, 675] and of a theorem of R. Baer [Amer. J. Math. **61** (1939), 1–44] on the lattice-isomorphisms of abelian groups.

A. Kertész (Debrecen)

5656:

Nunke, R. J. On the extensions of a torsion module. Pacific J. Math. **10** (1960), 597–606.

This paper concerns the structure of $\mathrm{Ext}_R{}^1(A,T)$, where A is a torsion-free and T is a torsion module over a Dedekind ring R. A basic submodule of T is a pure submodule B of T such that B is a direct sum of cyclic modules and T/B is divisible. Then $\mathrm{Ext}_R{}^1(A,T)=\mathrm{Ext}_R{}^1(A,B)$. Let P be a prime ideal of R, and assume from now on that T is a P-primary module. Let $r_P(P^nB)$ be the dimension over R/P of the submodule of P^nB annihilated by P. Then $\aleph=\inf_n r_P(P^nB)$ is called the critical number of T, after Szele. It is an invariant of T and is either 0 or infinite. If $\aleph=0$, then $\mathrm{Ext}_R{}^1(A,T)=0$; while if $\aleph$ is infinite, then $\mathrm{Ext}_R{}^1(A,T)\simeq\mathrm{Ext}_R{}^1(A,M)$, where M is a direct sum of $\aleph$ copies of $\sum_n R/P^n$. Let M^* be the P-adic completion of M, let Q be the quotient field of R, let $|R|$ be the cardinal number of elements of R, and let $r(M^*)$ be the rank of M^* (i.e., $r(M^*)$ is the vector space dimension over Q of $M^*\otimes Q$). Then $r(M^*)=(\aleph|R|)^{\aleph_0}$. If $\aleph$ is infinite, then the rank of $\mathrm{Ext}_R{}^1(Q,T)$ is equal to $r(M^*)$.

E. Matlis (Evanston, Ill.)

5657:

Fried, E. Über eine Verallgemeinerung der Auflösbarkeit von Gleichungen. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **2** (1959), 67–71.

Let L be a cyclic field extension of K. If $n=[L:K]$ is prime to the characteristic and the nth roots of 1 are in K, then $L=K[\alpha]$ with α a root of λ^n-a; if n equals the characteristic, then $L=K[\alpha]$ with α a root of $\lambda^n-\lambda-a$. In both of these cases, $(\alpha-1)^{-1}$ and its conjugates form a normal basis of L over K. An element is said to be expressible by quasiradicals if it lies in a tower of fields, each obtained from the preceding by adjoining an α of one of the two types above; then an equation is solvable by quasiradicals if and only if its Galois group is solvable.

D. Zelinsky (Evanston, Ill.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

See also 5650, 5697, 5698.

5658:

Govorov, V. E. Algebras freely generated by finite amalgams. Mat. Sb. (N.S.) **50** (**92**) (1960), 241–246. (Russian)

The author establishes the following generalization of Žukov's theorem [Mat. Sb. (N.S.) **26** (**68**) (1950), 471–478; MR **12**, 238]: (I) Let G be the (non-associative) algebra over a field F freely generated by an amalgam D of algebras [cf. Dididze, ibid. **43** (**85**) (1957), 379–396; MR **20** #3198]. If G is the homomorphic image of a finitely generated free algebra S, then S has a set of free generators whose images belong to D. Next, let U be a given algebra and consider the class $\mathfrak{U}$ of U-algebras (i.e., algebras with U as subalgebra). This class contains free U-algebras, and for any set of U-algebras, we can form their free sum, amalgamating U [cf. Dididze, loc. cit.]. Now the author proves: (II) If G is the free sum of the U-algebras $A_1, \cdots, A_k$ (amalgamating U) and G is the homomorphic image of the free U-algebra S on a finite generating set, then this generating set may be so chosen that each free generator maps into some A_i.

F. M. Cohn (Manchester)

5659:

Leger, G.; Tôgô, S. Characteristically nilpotent Lie algebras. Duke Math. J. **26** (1959), 623–628.

A Lie algebra, all of whose derivations are nilpotent, is called characteristically nilpotent [Dixmier and Lister, Proc. Amer. Math. Soc. **8** (1957), 155–158; MR **18**, 659]. The authors present some results concerning the structure of these algebras. Sample theorems: A Lie algebra is characteristically nilpotent if and only if it is not one-dimensional and its derivation algebra is nilpotent. A Lie algebra over an algebraically closed field of characteristic zero is characteristically nilpotent if and only if all of its semi-simple automorphisms are of finite order.

K. deLeeuw (Princeton, N.J.)

HOMOLOGICAL ALGEBRA

See also 5656, 5977.

5660:

Swan, Richard G. Projective modules over finite groups. Bull. Amer. Math. Soc. **65** (1959), 365–367.

The author announces the following result. Theorem 1: Let π be any finite group. Then any finitely generated projective module over the integral group ring $Z\pi$ is the direct sum of a free module and an ideal I of $Z\pi$. Furthermore, I can be chosen so that its index in $Z\pi$ (necessarily finite if $I\neq 0$) is prime to any given integer.

The author's techniques yield new results about the projective class group $C(Z\pi)$, which is defined by grouping projective modules over $Z\pi$ into equivalence classes, setting $P_1\sim P_2$ if we have $P_1+F_1\cong P_2+F_2$ with F_1, F_2 free. It follows from theorem 1 that $C(Z\pi)$ is finite. In order to state theorem 4, one introduces the homomorphism $i(\pi,\pi'): C(Z\pi)\to C(Z\pi')$ induced by the inclusion of a subgroup π' in π. One calls π' hyperelementary if it is a split extension in which the kernel is cyclic and the cokernel is a p-group; one calls it elementary if this extension is a direct product. [Cf. Brauer and Tate, Ann. of Math. (2) **62** (1955), 1–7; MR **16**, 1087].

Theorem 4: Let n be the common order of π. Let d be the greatest common divisor of n and $\phi(n)$, where ϕ is Euler's ϕ-function. Let $\alpha\in C(Z\pi)$. Then: (1) if $i(\pi,\pi')\alpha=0$ for all cyclic subgroups π' of π, then $n\alpha=0$; (2) if $i(\pi,\pi')\alpha=0$ for all elementary subgroups π' of π, then $d\alpha=0$; (3) if $i(\pi,\pi')\alpha=0$ for all hyperelementary subgroups π' of π, then $\alpha=0$.

The other theorems given are subsidiary results used in the proofs of theorems 1 and 4. The author's methods are interesting, but are only sketched briefly.

J. F. Adams (Cambridge, England)

5661:

MacLane, Saunders. Triple torsion products and multiple Künneth formulas. Math. Ann. **140** (1960), 51–64.

The Künneth formula for the homology of a tensor product of three factors, obtained by iterating that for two factors, is not formally symmetric in the three factors. The author remedies this by introducing new functors of three arguments, which are of intrinsic interest. Write H_i for the homology of a chain complex K_i of an abelian group, and H for that of $K_1\otimes K_2\otimes K_3$. Assuming that at least two of the K_i have no torsion, one has in H a series of subgroups with quotients naturally isomorphic to $Q_1=\mathrm{Tor}(\mathrm{Tor}(H_1,H_2),H_3)$, $Q_2=\mathrm{Tor}(H_1\otimes H_2,H_3)$, $Q_3=\mathrm{Tor}(H_1,H_2)\otimes H_3$, and $Q_4=H_1\otimes H_2\otimes H_3$; this decomposition splits, but the isomorphism of H to the direct sum of the quotients is not natural. Although Q_1 is known to be symmetric in the H_i, it is not symmetrically defined. A functor $\mathrm{Tor}(A_1,A_2,A_3)$ of graded groups is defined by generators and relations in a symmetric manner, similar to that by which Eilenberg and the author defined $\mathrm{Tor}(A_1,A_2)$ [Ann. of Math (2) **60** (1954), 49–139; MR **16**, 391]. Now Q_1 is naturally isomorphic to $\mathrm{Tor}(H_1,H_2,H_3)$. A second functor $\mathrm{Trip}(A_1,A_2,A_3)$ is defined symmetrically by imposing on the direct sum T of the three cyclic permutations of $\mathrm{Tor}(A_1,A_2)\otimes A_3$ a 'Jacobi identity'. The new triple Künneth formula now asserts that H has a chain of subgroups with quotients naturally isomorphic to $\mathrm{Tor}(H_1,H_2,H_3)$, $\mathrm{Trip}(H_1,H_2,H_3)$, and $H_1\otimes H_2\otimes H_3$.

As expected, Trip(A_1, A_2, A_3) has a chain with quotients Tor$(A_1 \otimes A_2, A_3)$ and Tor$(A_1, A_2) \otimes A_3$, and this decomposition splits. But, it is shown, Trip(A_1, A_2, A_3) is not isomorphic to the direct sum of the two quotients under any natural isomorphism. It is noted that Trip can be described as the quotient of the functor T mentioned above by a functor S with a definition reminiscent of that of Tor(A_1, A_2, A_3). The question of whether Trip is a composite of functors of two variables is left open, as is that of what happens for more than three factors, or for modules over rings other than the integers.

R. C. Lyndon (London)

5662:

Hattori, Akira. On fundamental exact sequences. J. Math. Soc. Japan **12** (1960), 65–80.

The Hochschild-Serre exact sequences for groups and Lie algebras [Trans. Amer. Math. Soc. **74** (1953), 110–134; Ann. of Math. (2) **57** (1953), 591–603; MR **14**, 619, 943] as well as generalizations by Adamson, Nakayama and Hirata are all shown (without the use of spectral sequences) to be special cases of an exact sequence involving relative cohomology in the sense of Hochschild [Trans. Amer. Math. Soc. **82** (1956), 246–269; MR **18**, 278]. If R is a ring and S a subring and if A and B are left R-modules, then directly from the definitions of the functors involved one defines restriction and lift mappings $\rho: \mathrm{Ext}_R{}^n(A, B) \to \mathrm{Ext}_S{}^n(A, B)$ and $\lambda: \mathrm{Ext}_{R,S}{}^n(A, B) \to \mathrm{Ext}_R{}^n(A, B)$ with $\rho\lambda = 0$. Under suitable further hypotheses, the author defines the subgroup $[\mathrm{Ext}_S{}^n(A, B)]^R$ of stable elements in $\mathrm{Ext}_S{}^n(A, B)$ and the transgression τ from this subgroup to $\mathrm{Ext}_{R,S}^{n+1}(A, B)$. The theorem states that if R is left S-projective and suitable lower dimensional Ext's vanish, then

$$0 \to \mathrm{Ext}_{R,S}{}^n(A, B) \xrightarrow{\lambda} \mathrm{Ext}_R{}^n(A, B) \xrightarrow{\rho} [\mathrm{Ext}_S{}^n(A, B)]^R \xrightarrow{\tau} \mathrm{Ext}_{R,S}^{n+1}(A, B) \xrightarrow{\lambda} \mathrm{Ext}_R^{n+1}(A, B)$$

is exact.

D. Zelinsky (Evanston, Ill.)

5663:

Leslie, Joshua. Modules simpliciaux sur une algèbre simpliciale. C. R. Acad. Sci. Paris **251** (1960), 22–23, 190–191.

The author continues his discussion of modules over a simplicial algebra (i.e., algebra complex) [C. R. Acad. Sci. Paris **248** (1959), 2692–2694; MR **21** #3836]. He introduces in addition to the exact structure the weaker abelian structure in which extensions are twisted products, and shows that with respect to either structure every object is a quotient of a projective. He then studies Ext$(X, K(\pi, n))$, first in the special case of the category of abelian group complexes, and then in the case that the algebra is the integer group algebra of a group complex. For the former he produces a spectral sequence of the Künneth type, for the latter an exact sequence involving the augmentation ideal. There are several troublesome misprints, e.g., $\mathrm{Ext}_Z{}^p$ for $\mathrm{Ext}_{\mathbf{Z}}{}^p$ on the last line of page 23.

A. Heller (Urbana, Ill.)

GROUPS AND GENERALIZATIONS

See also 5932, 5962.

5664:

Rigby, J. F. Monomial groups with respect to a basic Abelian group. Proc. London Math. Soc. (3) **10** (1960), 239–252.

A monomial matrix M contains n^2-n zeros; the remaining n elements (one in each row, one in each column) come from a (multiplicative) abelian group $A\{1, a, b, \cdots\}$. Matrix multiplication is defined in the obvious way $[0+a=a]$. A monomial group G is a group of monomial matrices of determinant ± 1, e.g., $\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&a\\b&0\end{pmatrix}\right\}$, $ab=1$. The permutation group $P=P(G)$ associated with G contains (by definition) the matrices which result when the nonzero elements of the matrices of G are replaced by 1. The diagonal part $H=H(G)$ is the kernel of the homomorphism $G \to P(G)$.

When $P(G)$ contains the alternating group, the author determines all possible monomial groups (A being abelian) rather explicitly, to within transformation by a monomial matrix. Typical results are the following. (Example 1) Suppose that $n=6$, and that $P(G)$ is the alternating group. Then either G is the product of $H(G)$ and a permutation group, or else $G=\{H, x, y, z\}$, where

$$x = \mathrm{diag}(M_3, b, b^2, b^{-3}),$$
$$y = \mathrm{diag}(b, b^2, M_3, b^{-3}),$$
$$z = \mathrm{diag}(b^2, b, b^{-3}, M_3), \quad M_3 = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}.$$

(Example 2) Let $H(G)$ be the identity, $P(G)$ the alternating group, $n=5$. Then either

$$G = \{\mathrm{diag}(b, b^2, M_3), \mathrm{diag}(M_3, b, b^2), b^3 = 1\}$$

or else $G=P(G)$.

J. L. Brenner (Madison, Wis.)

5665:

Fridman, M. A. A question concerning completely regular operations on a class of groups. Uspehi Mat. Nauk **14** (1959), no. 3 (87), 181–183. (Russian)

In recent years several types of associative "regular" operations on groups, apart from the direct and the free product, have been discovered and discussed [see, e.g., O. N. Golovin, Mat. Sb. (N.S.) **27** (**69**) (1950), 427–454; MR **12**, 672; S. Moran, Proc. London Math. Soc. (3) **6** (1956), 581–596; MR **20** #3908; R. R. Struik, Trans. Amer. Math. Soc. **81** (1956), 425–452; MR **17**, 1051; M. Benado, C. R. Acad. Sci. Paris **243** (1956), 1092–1093; MR **18**, 279; M. A. Fridman, Dokl. Akad. Nauk SSSR **109** (1956), 710–712; MR **18**, 279]. A. I. Mal'cev has raised the question whether there exist such completely regular operations that have the additional property of being hereditary on subgroups: suppose that $G=\prod^* G_\alpha$ is a completely regular product of groups G_α, that a subgroup H_α of G_α is chosen for each α, and that H is the group generated by the H_α. Is then $H=\prod^* H_\alpha$? In this note the author proves that as far as his semi-commutative multiplications [loc. cit.] are concerned the direct and the free product are the only ones with this additional property.

K. A. Hirsch (St. Louis, Mo.)

5666:

Piccard, Sophie. Les groupes quasi libres. C. R. Acad. Sci. Paris **250** (1960), 3260–3262.

The author studies quasi-free groups, a generalization of the notion of free group. A group is called quasi-free if all generating relations are quasi-trivial. This means, for a generating relation of the form $a_1a_2 \cdots a_n = 1$, that every generator a occurs the same number of times as its inverse a^{-1}. The number of generators and generating relations may be infinite. It is shown by examples that the class of quasi-free groups is quite extensive.

With every quasi-free group the author associates a collection of abelian groups which form a lattice with respect to a divisibility relation. This lattice of abelian groups may be used to study the structure of the quasi-free group. The author states without proof that many of the results of her paper on free groups in Ann. Sci. École Norm. Sup. **76** (1959), 1–58 [MR **21** #4969] may be extended to quasi-free groups. In particular, every quasi-free group with infinitely many generators has an uncountable infinity of normal subgroups.

O. Frink (Dublin)

5667:

X **Jakubík, Ján. Über eine Eigenschaft von l-Gruppen.** Časopis Pěst. Mat. **85** (1960), 51–59. (Russian. Slovak and German summaries)

An l-subgroup of a direct product of l-groups $\{G_\sigma\}$ is called a complete subdirect product if it contains the direct sum of the G_σ's. The paper under review supplements the work of Šik [Czechoslovak Math. J. **8** (**83**) (1958), 22–53; MR **20** #5810] on complete subdirect product of ordered groups. In his paper, Šik gives three conditions which together are necessary and sufficient for an l-group G to be isomorphic to a complete subdirect product of ordered groups. The author shows that the conditions of Šik are mutually independent, but that they can be replaced by two different conditions: (1) every $a \in G^+$ can be written in the form $x \cup y$ with $x \cap y = 0$, $x > 0$ and the interval $[0, x]$ a chain; (2) if $0 < z < a$ and $[0, z]$ is a chain, then the x, y in (1) can be chosen so that $x \geqq z$. It is observed that these conditions are pure lattice theoretic properties of G^+. Similarly, conditions for a complete l-group to be a complete subdirect product of ordered groups can be formulated as lattice properties of G^+. On the other hand, it is impossible to determine from the lattice structure of G^+ whether or not G admits a proper complete subdirect decomposition into l-groups.

R. S. Pierce (Seattle, Wash.)

5668:

Popp, Gerhard Christof. Beiträge zur Theorie der Zappaschen Produkte. Math. Nachr. **20** (1959), 303–328.

A Zappa product (also known by the names of Rédei and Szép and as "general product") of two groups A, B is a group $G = AB$ in which A, B intersect trivially. The author studies not so much G as the algebraic system consisting of A and B and the way in which they are combined to form G. He defines and investigates such notions as subproducts, normal subproducts, factor products. An analogy between direct products and abelian groups is used to define a certain normal subproduct which is analogous to the derived group, and a notion of "decomposability" ("Zerlegbarkeit") which corresponds to solubility. Thus it is shown that a product of two finite abelian groups is decomposable {such a product is known to be metabelian, cf. N. Itô, Math. Z. **62** (1955), 400–401; MR **17**, 125}. If a group A is embedded in the group P of permutations of its elements by means of the regular representation by right multiplications, then P becomes the Zappa product of A and the stabilizer of the unit element; and the same is true of every subgroup of P that contains A. To every Zappa product there are Zappa products of free groups of which the given product is a factor product. Finally the Zappa product is considered as a homomorphic image of the free product, and the kernel of the homomorphism is investigated. It turns out to be a free group of rank determined by the factors only, independently of the way in which they are combined in the Zappa product; and the automorphism class group of this free group contains an isomorphic copy of every such Zappa product.

B. H. Neumann (Manchester)

5669:

Sąsiada, E. On two problems concerning endomorphism groups. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **2** (1959), 65–66.

In a very succinct fashion an example is constructed which answers yes and no respectively to two of the problems of L. Fuchs on abelian groups: Do groups G and H exist whose endomorphism rings are not isomorphic, but whose endomorphism groups are isomorphic? Is the endomorphism group End (G) indecomposable if G is indecomposable?

D. K. Harrison (Philadelphia, Pa.)

5670:

Baumslag, Gilbert; Blackburn, Norman. Direct summands of unrestricted direct sums of abelian groups. Arch. Math. **10** (1959), 403–408.

The following four theorems are proved. The restricted direct sum G of a set of groups G_i $(i = 1, 2, \cdots)$ is a direct summand of the unrestricted direct sum G^* if and only if the exponents of all but a finite number of the reduced parts of the groups G_i are bounded. A torsion direct summand of the unrestricted direct sum of p-groups of finite exponent is itself of finite exponent. A torsion direct summand of the unrestricted direct sum of cyclic p-groups of strictly increasing orders is finite. A non-trivial torsion-free direct summand of an unrestricted direct sum of (countably many) finite p-groups has the cardinality of the continuum. {The authors ask whether the unrestricted direct sum of cyclic groups of orders $p, p^2, \cdots$ has a non-trivial torsion-free direct summand; the answer to this is yes. The p-adic integers, as an inverse limit, form a subgroup of this group, and since each group in the inverse limit is a pure subgroup and thus is a direct summand, it can be proved that the inverse limit itself is a direct summand.}

D. K. Harrison (Philadelphia, Pa.)

5671:

Król, M.; Sąsiada, E. The complete direct sums of torsion-free abelian groups of rank 1 which are separable. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **8** (1960), 1–2. (Russian summary, unbound insert)

Problem 25 in the reviewer's book *Abelian groups* [Akadémiai Kiadó, Budapest, 1958; Pergamon, New York, 1960; MR **21** #5672; **22** #2644] is solved by

giving a complete description of the groups in the title in terms of the types of the components. Necessary and sufficient conditions are: (1) if χ_t are characteristics (=heights) of elements of components of different types, then any infinite intersection $\bigcap \chi_t$ is equal to the intersection of a finite subset $\chi_{t_1}, \cdots, \chi_{t_n}$; (2) each type with infinitely many natural integers occurs but a finite number of times among the components. *L. Fuchs* (Budapest)

5672:

Maranda, J.-M. On pure subgroups of abelian groups. Arch. Math. **11** (1960), 1–13.

Let G be an abelian group (for simplicity rather than a module over a principal ideal domain). For S a set of primes, a subgroup H of G is called S-pure if $H \cap p^n \cdot G = p^n \cdot H$ for all $p \in S$ and all integers n. The group G is called S-projective if $\mathrm{Hom}(G, A) \to \mathrm{Hom}(G, A/B)$ is onto for all groups A and S-pure subgroups B of A. G is called S-injective if $\mathrm{Hom}(A, G) \to \mathrm{Hom}(B, G)$ is onto for all such A and B. Then G is S-projective if and only if it is a direct summand whenever it is the image of a homomorphism whose kernel is S-pure; this in turn is equivalent to G being a direct sum of infinite groups and p-power cyclic groups for p's in S. G is S-injective if and only if it is a direct summand whenever it is an S-pure subgroup; this in turn is equivalent to G being a direct summand of a direct product of p-power cyclic groups and $Z(p^\infty)$ groups for p's in S. [Although the author does not develop an "S-Ext" theory, this can be done.] The author calls G an S-essential extension of an S-pure subgroup H if $L=0$ whenever $L \cap H = 0$ and $(L+H)/L$ is S-pure in G/L. It is established that every group has a maximal S-essential extension and that these maximal extensions are exactly the S-injective groups. A group is called S-algebraically compact if it is a direct product of a divisible group and of groups D_p, with $p \in S$, where each D_p is complete and Hausdorff in the metric defined by taking the $p^n \cdot D_p$ as spheres about the identity. Then any group is an S-pure, dense subgroup of some S-algebraically compact group. *D. K. Harrison* (Philadelphia, Pa.)

5673:

Robinson, Abraham; Zakon, Elias. Elementary properties of ordered abelian groups. Trans. Amer. Math. Soc. **96** (1960), 222–236.

Let M be an ordered abelian group. If $x-y=nz$ for x, y, z in M and a positive integer n, then x is congruent to y modulo n. Let $[n]M$ be the maximum finite number of elements in M which are mutually incongruent modulo n, or ∞ if such a finite number does not exist. M is regularly discrete if M is discretely ordered and such that $[p]M=p$ for every prime p, and M is regularly dense if for every positive integer n and each pair of elements $a<b$ in M, there exists $y \in M$ such that $a<ny<b$. M is regularly ordered if M is regularly discrete or regularly dense. The authors obtain a complete classification of regularly ordered abelian groups by their elementary properties (i.e., properties that can be formalized in the lower predicate calculus). It is shown that all regularly discrete groups are elementarily equivalent (i.e., have all their elementary properties in common), and that two regularly dense groups A and B are elementarily equivalent if and only if $[p]A=[p]B$ for every prime p. It follows from some results that are to be published elsewhere that each archimedean ordered group is regularly ordered, and that every regularly ordered abelian group is elementarily equivalent to an archimedean ordered group.

The method of proof depends upon the concept of model completeness and the process of adjoining definable predicates in the capacity of new atomic predicates, and simultaneously, adjoining their definitions as new axioms. In addition to the above results, these elegant methods of proof are used to prove the completeness of certain sets of axioms for ordered sets. For example, complete sets of axioms for the following types of ordered sets are given: densely ordered with no first and no last element; densely ordered with a first but no last element; discretely ordered with no first and no last element; discretely ordered with a first but no last element; infinite discretely ordered with a first and a last element. *P. F. Conrad* (New Orleans, La.)

5674:

Feit, Walter; Hall, Marshall, Jr.; Thompson, John G. Finite groups in which the centralizer of any non-identity element is nilpotent. Math. Z. **74** (1960), 1–17.

The following theorem is established. If a finite insoluble group is such that the centralizer of every non-identity element is nilpotent, then the group must have even order. This generalizes an earlier result of M. Suzuki [Proc. Amer. Math. Soc. 8 (1957), 686–695; MR **19**, 248], which differs from the present theorem in having "nilpotent" replaced by "abelian". The present proof uses the method of reductio ad absurdum on a huge scale. Call G a CN-group if every $x \neq 1$ in G has nilpotent centralizer. If the theorem is false, it quickly follows that there exists a CN-group G which is simple, of odd order, and in which every proper subgroup is soluble. The proof then consists in showing that such a group G cannot exist. The first half of the argument is purely group-theoretical. It yields the information that in a group G, of the type under consideration, two distinct Sylow subgroups always intersect trivially and hence that G is partitioned by its maximal nilpotent subgroups. (A subsidiary result (lemma 1.8) gives insight into the structure of soluble CN-groups of odd order.) The second half of the proof is character-theoretical. If $\{H_1, \cdots, H_r\}$ is a complete set of representatives for the conjugacy classes of maximal nilpotent subgroups of G, then it is shown that every non-principal irreducible character of G can be assigned to precisely one of $H_1, \cdots, H_r$, and that the set of all those assigned to H_i is in one-one correspondence with the non-principal irreducible characters of H_i. The explicit form of these linkages gives rise to inequalities from which the desired contradiction results.

(Misprint: All references in the proofs of Lemmas 1.8 and 1.9 to item 2 of the bibliography should be to item 3.) *K. Gruenberg* (London)

5675:

Dlab, Vlastimil. Note on a problem concerning Frattini subgroups. Časopis Pěst. Mat. **85** (1960), 87–90. (Czech. Russian and English summaries)

Let the group G be the direct product of its subgroups G_δ, with δ ranging over an index set Δ. It is proved that the Frattini subgroup $\Phi(G)$ of G is the direct product of the $\Phi(G_\delta)$ if one of the following conditions holds: (a) G is abelian; (b) $\Phi(G_\delta)$ is finitely generated for every

$\delta \in \Delta$; (c) G_δ is finitely generated for every $\delta \in \Delta$; (d) G is finitely generated. *R. Kochendörffer* (Rostock)

5676:

Dlab, Vlastimil. The Frattini subgroups of abelian groups. Czechoslovak Math. J. **10** (85) (1960), 1–16. (Russian summary)

The author determines the Frattini subgroup of an arbitrary abelian group; the book *Abelian groups* by L. Fuchs [Akadémiai Kiadó, Budapest, 1958; MR **21** #5672], in which the same result occurs as an exercise, came to the author's notice only after his paper had gone to press. Every abelian group is shown to be the Frattini subgroup of an abelian group, and the latter can be chosen minimal and is then unique up to isomorphism. In this way ascending Frattini series of abelian groups can be formed; if the terms are taken to be minimal, then such a series becomes stationary after ω steps when it arrives at the divisible closure of the first term, unless the first term is already divisible (when the series is stationary from the start). Descending Frattini series of abelian groups are also considered, and applied to study sets of generators of abelian groups. The author remarks that Satz 18* on p. 122 of the book *Gruppentheorie* by Wilhelm Specht [Springer, Berlin, 1956; MR **18**, 189] is false. *B. H. Neumann* (Manchester)

5677:

Carter, R. W. On a class of finite soluble groups. Proc. London Math. Soc. (3) **9** (1959), 623–640.

A subgroup H of a finite soluble group G is called abnormal if, for every subgroup K of G, $H \subset K$ and $H \subset g^{-1}Kg$ imply that $g \in K$. For instance, the normaliser of a Sylow subgroup is abnormal. Every abnormal subgroup contains a system normaliser; and every subgroup containing a subnormal subgroup is subnormal. This suggests studying those soluble groups in which the system normalisers are abnormal, and this is the class mentioned in the title.

A group G belongs to the class if and only if its system normalisers are their own normalisers. For groups G in the class the following conditions on a subgroup H are equivalent: (i) H is abnormal; (ii) H contains a system normaliser; (iii) H can be joined to G by a chain of subgroups each maximal but not normal in the following one; (iv) every system normaliser of H is a system normaliser of G. The class is closed under formation of direct products and homomorphic images, but not of subgroups. It contains all groups of nilpotent length 2. Any soluble group G has a unique normal subgroup N minimal with respect to G/N belonging to the class.

Other theorems are of "covering and avoidance" type; or concern conditions under which the intersections with H of a Sylow system of G form a Sylow system of H. Results of both these types are tools used continually in the proofs even of theorems apparently unconnected with them. *Graham Higman* (Chicago, Ill.)

5678:

Baer, Reinhold. Kriterien für die Endlichkeit von Gruppen. Jber. Deutsch. Math. Verein. **63**, Abt. 1, 53–77 (1960).

A survey of conditions for a group to be finite, divided into five sections: A. Conditions for an abelian group to be finite, in terms of its character group. B. Conditions on the finiteness of the abelian parts of a group, which imply the finiteness of the whole group. For instance, an FC group is finite if its abelian subgroups are finite. C. Conditions which make a finitely generated group finite, mostly in terms of subnormal structure. D. Conditions on the endomorphisms of a group which make it finite. It suffices that there be only a finite number of endomorphisms, or, for a torsion group, of automorphisms. E. Conditions on the Sylow structure which imply that the group is finite. For instance, a torsion group is finite if it has p-elements for only a finite number of primes p; and for each p, at least one finite Sylow subgroup, and at least one Sylow subgroup with only a finite number of conjugates. *Graham Higman* (Chicago, Ill.)

5679:

Černikov, S. N. Finiteness conditions in the general theory of groups. Uspehi Mat. Nauk **14** (1959), no. 5 (89), 45–96. (Russian)

This paper is a companion piece to the paper by B. I. Plotkin in Uspehi Mat. Nauk **13** (1958), no. 4 (82), 89–172 [MR **21** #686]. It surveys, after a brief historical and systematic introduction, the progress made during the last two decades. The bibliography contains 80 titles, all but 14 by Russian authors. Not fewer than 26 papers are by Černikov himself, an outward sign of the author's contribution to our knowledge in the field. The finiteness conditions that are treated here in much detail are the minimal condition for subgroups, for normal subgroups, for abelian subgroups, and for subgroups of finite index. (For an account of the corresponding maximal conditions one should consult the paper by Plotkin [loc. cit.].) In addition there are theorems on socle-finite groups, i.e., groups in which the factor-group of every finite normal subgroup has a finite non-trivial socle, on groups with finite Sylow subgroups (where a number of new results of the author are announced, partly with and partly without proofs), on groups with a finite number of conjugacy classes and on groups with a finite number of elements in each conjugacy class. As is usual in these valuable Russian surveys, the paper lists a number of important unsolved problems. *K. A. Hirsch* (St. Louis, Mo.)

5680:

Gorenstein, Daniel. On the structure of certain solvable groups. Proc. Sympos. Pure Math., Vol. 1, pp. 15–21. American Mathematical Society, Providence, R.I., 1959.

A survey of the results to date on the structure of finite groups of the form ABA. *Graham Higman* (Chicago, Ill.)

5681:

Pazderski, Gerhard. Die Ordnungen, zu denen nur Gruppen mit gegebener Eigenschaft gehören. Arch. Math. **10** (1959), 331–343.

Gesucht wird eine zahlentheoretische Charakterisierung aller derjenigen Ordnungszahlen n, zu denen nur Gruppen der Eigenschaft $\mathfrak{e}$ gehören. Z.B. werden Gruppen mit der Eigenschaft zyklisch zu sein durch $(n, \varphi(n)) = 1$ charakterisiert. Das Problem wird für die folgenden Eigenschaften $\mathfrak{e}$ gelöst: (1) Nilpotent zu sein; (2) einen geord-

neten Sylowturm zu haben; (3) überauflösbar zu sein; (4) eine zyklische Kommutatorgruppe zu besitzen; (5) metazyklisch zu sein. Als wesentliches Beweishilfsmittel wird herangezogen, dass sich diese Eigenschaften auf die Untergruppen übertragen. (1) wird durch $(n, \psi(n))=1$ gelöst. Mit $n=p_1^{a_1}\cdots p_r^{a_r}$ wird (2), durch

$$(p_i^{a_i}\cdots p_r^{a_r}, \psi(p_i^{a_i}))=1$$

für $i=1, 2, \cdots, r$, gelöst. Für die übrigen Fälle lassen sich die Bedingungen nur etwas komplizierter formulieren; es sei auf die Originalarbeit verwiesen.

J. J. Burckhardt (Zürich)

5682:

Čarin, V. S. On the theory of locally nilpotent groups. Amer. Math. Soc. Transl. (2) **15** (1960), 33–54.

Translation of Mat. Sb. (N.S.) **29** (71) (1951), 433–454 [MR **14**, 1059].

5683:

Kertész, A. On radical-free rings of endomorphisms. Acta Univ. Debrecen **5** (1958), 159–161 (1959). (Hungarian summary)

The author considers the ring $E(G)$ of all endomorphisms of an abelian group G and raises the question of when the Jacobson radical of $E(G)$ is equal to zero. The following answer is given for torsion groups (every element of finite order): The ring of endomorphisms of an abelian torsion group G is radical-free if and only if G is an elementary abelian group, i.e., if it is the direct sum of groups of prime order.

C. E. Rickart (New Haven, Conn.)

5684:

Nagao, Hirosi. On GL(2, $K[x]$). J. Inst. Polytech. Osaka City Univ. Ser. A **10** (1959), 117–121.

Let G be the group of nonsingular 2 by 2 matrices with entries in $K[x]$, K a field. The author first gives an explicit presentation of G as a factor group of the free product of GL(2, K) with the group of all matrices of the form $I+f(x)e_{12}$, $f(x)$ in $K[x]$. From this he deduces that G is not finitely generated. He next shows that the subgroup of G generated by the subgroups $I+K[x]e_{12}$ and $I+K[x]e_{21}$ is isomorphic to the free product of these groups. It is further shown that if a is in K and p is the characteristic of K, then $I+ae_{21}$ and $I+xe_{12}$ generate the free product of two cyclic groups of order p if a cyclic of order 0 is understood to mean an infinite cyclic group. Finally, the author proves that the free product of two free abelian groups, or elementary abelian groups with the same exponent, has an isomorphic matrix representation of degree 2 over a suitable field. The last two results extend and complement the work of Chang, Jennings and Ree [Canad. J. Math. **10** (1958), 279–284; MR **20** #906].

A. Rosenberg (Berkeley, Calif.)

5685:

Schweizer, B.; Sklar, A. The algebra of functions. Math. Ann. **139**, 366–382 (1960).

The authors develop systematically the consequences of a set of five axioms designed to describe the algebraic behavior of real-valued functions of a real variable, with different domains of definition, under any one of the three operations of addition, multiplication, and composition (substitution). These axioms are equivalent to a set given by K. Menger [*The axiomatic method* (Berkeley symposium, 1957), pp. 454–473, North-Holland, Amsterdam, 1959; MR **22** #2540]. They describe a system which is a partially ordered semigroup with identity, in which every element has a kind of generalized inverse called a neutralizer, having a certain maximality property relative to the partial ordering. The order relation $f\subset g$ has the interpretation that f is a restriction of the function g to a smaller domain of definition.

Three kinds of inverses, called left and right neutralizers, invertors, and subinverses, are defined and studied. It is shown that the set of all invertible elements is closed under restriction, subinversion, and the semigroup operation. Other notions introduced are those of idempotent, constant, subconstant, annihilator, and null element. It is shown that constants are idempotent and form a left ideal, that subconstants form a two-sided ideal, and that the null element, if it exists, is unique and an annihilator.

O. Frink (Dublin)

5686:

Villamayor, Orlando E. On the semisimplicity of group algebras. II. Proc. Amer. Math. Soc. **10** (1959), 27–31.

In a previous note [same Proc. **9** (1958), 621–627; MR **20** #5224] the author found a partial solution to the problem of the semisimplicity of group algebras by proving that, if K is a semisimple commutative algebra over the rationals and G is a group which is locally finite over the center, then the group algebra $K(G)$ is semisimple. In this note the result is extended to a class of groups called GSN-groups, which includes the class of SN-groups [A. G. Kurosh, *The theory of groups*, vol. II, Chelsea, New York, 1956; MR **18**, 188]. A group is an SN-group if it has a $\mathfrak{C}$-normal system, where $\mathfrak{C}$ is the class of all commutative groups, and is a GSN-group if it has a $\mathfrak{C}^*$-normal system, where $\mathfrak{C}^*$ is the class of all groups which are locally finite over their center.

G. L. Walker (Providence, R.I.)

5687:

Suzuki, Michio. Applications of group characters. Proc. Sympos. Pure Math., Vol. 1, pp. 88–99. American Mathematical Society, Providence, R.I., 1959.

The paper is concerned with applications of the classical (non-modular) theory of group characters. Some of the results mentioned are attributed to R. Brauer, or jointly to him and to the author.

Let G be a finite group of order g. The irreducible characters generate, additively, a free abelian group M_G of generalized characters. Two elements ρ, σ of G are called equivalent if $\varphi(\rho)=0$ for $\varphi\in M_G$ implies that $\varphi(\sigma)=0$. It is shown that ρ and σ are equivalent if and only if σ is conjugate to some power ρ^k of ρ, where $(k, g)=1$. This equivalence relation induces a partition of G into equivalence classes. A subset C of G is called closed if it is the join of one or several such classes. The author considers subgroups $M_G(C)$ of M_G consisting of those generalized characters which vanish on the complement of C. In particular, he discusses their properties with respect to the inner-product metric

$$\langle\phi, \psi\rangle_G=(1/g)\sum_{\sigma\in G}\phi(\sigma)\psi(\sigma^{-1}).$$

These ideas are then applied to characters of G which are induced by characters of a subgroup H. Let D be a closed set of H satisfying the following conditions: (1) if two elements of D are conjugate in G, then they are conjugate in H; and (2) if $\rho \in D$, then the centralizer of ρ in G is contained in H. Under these assumptions $\langle\theta^*, \theta^*\rangle_G = \langle\theta, \theta\rangle_H$ for any θ in $M_H(D)$, where θ^* is the character of G induced by θ. Thus if θ is a linear combination of only two or three irreducible characters, so also is θ^*. The author indicates several applications of this theory without giving details of the computations in all cases. The main result which can be obtained by this method is the following theorem: if a 2-Sylow subgroup of a finite group G is a generalized quaternion group, then G contains a normal subgroup N of odd order, and G/N contains only one element of order 2.

W. Ledermann (Manchester)

5688:

Sade, Albert. Quasigroupes parastrophiques. Expressions et identités. Math. Nachr. **20** (1959), 73–106.

This paper is concerned with the same corpus of ideas as S. K. Stein, Trans. Amer. Math. Soc. **85** (1957), 228–256 [MR **20** #922]; but notation and nomenclature are different. The bibliography usefully supplements the extensive bibliography of Stein's paper.

Let $x \times y = z$ in a quasigroup $Q(\times)$ on the set E. Then the author writes $y * x = z$, $z \ominus y = x$, $z \oplus x = y$, $x \odot z = y$, $y \otimes z = x$, and so defines the set of 6 parastrophic quasigroups $Q(\times)$, its conjoint $Q(*)$, its reciprocal $Q(\ominus)$, etc. Each corresponds not only to a permutation of x, y, z but also to a permutation of $\times * \ominus \oplus \odot \otimes$; e.g., $Q(*)$ rewritten as $Q(\times')$ corresponds to the permutation $P* = (\times *)(\ominus \oplus)(\odot \otimes)$. An expression means a suitably bracketed word in the elements of E and the operational symbols; it is integral (entière) if $\times$ is the only operational symbol involved. Given an identity (i.e., identical equality between expressions), the corresponding identity in a parastrophic quasigroup is obtained by applying the appropriate permutation to the operational symbols and then if necessary applying rules of simplification. (In contrast, Stein by 'identity' meant integral identity, so that in his terminology there were 'identities' corresponding parastrophically to 'constraints not expressible as identities'.) Among the topics studied are: self-conjoint, self-reciprocal and self-parastrophic identities; conditions under which further rules of simplification apply, analogous to rules for adding and multiplying fractions in ordinary algebra; distributivity of other operations over one or more of the 6 parastrophic operations; 'the different forms of an identity' and 'proper equivalent forms'. {This last concept is puzzling. The equivalent identities $a \ominus b = a \oplus b$, $c \times b = b \times c$ are said to be 'trivially' or 'improperly' equivalent. Omitting for brevity the sign $\times$, the same is said of $xy = yx$, $a(bc) = (bc)a$; whereas by a definition obscure to the reviewer the identities (1) $(c \cdot ba)(cb) = a$ and (2) $b(cx \cdot cb) = x$, which are equivalent in quasigroups, are said to be 'properly equivalent'. The reviewer thinks that the author, and perhaps also Stein [loc. cit., p. 253], have overlooked that one passes from (2) to (1) immediately on replacing x by ba and cancelling b; the argument is reversible. Similarly for other examples given.}

The identity $a \times (b \times a) = b$ is ('properly') equivalent to $(a \times x) \times a = x$ and implies $a \times b = a \oplus b = a \otimes b$, $a * b = a \odot b = a \ominus b$. Hence for quasigroups with this property the parastrophic quasigroups coincide in threes. The author remarks that such quasigroups should prove interesting; he calls them demisymmetric by analogy with totally symmetric quasigroups [R. H. Bruck, ibid. **55** (1944), 19–52; MR **5**, 229] for which all six parastrophics coincide.

A sequence of elements $\{a_i\}$ is called a sequence of (i) right quotients relative to m if $a_i \times a_{i+1} = m$, (ii) right powers of x if $a_{i+1} = a_i \times x$, and (iii) right products if $a_i \times a_{i+1} = a_{i+2}$, with similar left definitions. Among typical results are: a sequence of one type is a sequence of the same or another type, perhaps reversed, in a parastrophic quasigroup; if $\{a_i\}$, $\{b_i\}$ are of the same type, conditions for $\{a_ib_i\}$ and $\{ka_i\}$ to be of the same type are found in terms of the entropic and self-distributive laws $ab \cdot cd = ac \cdot bd$, $ab \cdot c = ac \cdot bc$ ($\times$ understood).

Finally, there is a theorem connecting parastrophism and isotopy which 'renders precise' a remark by the reviewer [Proc. Edinburgh Math. Soc. (2) **7** (1945), 104–121; MR **7**, 4] {actually a remark concerning parastrophic linear algebras rather than quasigroups, a remark which was also made (p. 153) by R. H. Bruck, Trans. Amer. Math. Soc. **56** (1944), 141–199; MR **6**, 116}.

I. M. H. Etherington (Edinburgh)

5689:

Bruck, R. H. Sums of normal endomorphisms. Proc. Amer. Math. Soc. **10** (1959), 674–678.

The author continues the study of normal endomorphisms of loops begun in Chapter IV, § 4, of his book *A survey of binary systems* [Springer, Berlin, 1958; MR **20** #76] and pursued further in a comprehensive paper in Illinois J. Math. **4** (1960), 38–87 [MR **22** #4794]. The main theorem of the present paper is: "Let $(L, +)$ be the additive loop generated by the set of all normal endomorphisms of a loop G. A necessary and sufficient condition in order that $(L, +, \cdot)$ be an associative ring is that G be power-associative." If G is a group the definition of a normal endomorphism of a loop becomes equivalent to the usual (quite different) definition of a normal endomorphism of a group. Hence the theorem generalizes a result by N. Heerema stating that for a group G, $(L, +, \cdot)$ is an associative ring [Trans. Amer. Math. Soc. **84** (1957), 137–143; MR **18**, 559].

R. Artzy (Haifa)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 5659, 5900a–b.

5690:

Gluškov, V. M. The structure of locally compact groups and Hilbert's fifth problem. Amer. Math. Soc. Transl. (2) **15** (1960), 55–93.

Translation of Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2 (74), 3–41 [MR **21** #698].

5691:

Collins, H. S. Primitive idempotents in the semigroup of measures. Duke Math. J. **27** (1960), 397–400.

Let S be the semigroup of measures on a compact group G with Haar measure m. An idempotent μ in S is

called primitive in case $\mu S \mu$ contains μ and m as its only idempotents. Such μ are characterized here; a specimen result (part of theorem 1) states: μ is primitive if and only if H = carrier μ is a maximal proper closed subgroup of G. Theorem 2 takes up central primitive idempotents; for example, if μ is central then primitivity is equivalent to G/H being finite and of prime order.

J. G. Wendel (Ann Arbor, Mich.)

5692:

Stromberg, Karl. Probabilities on a compact group. Trans. Amer. Math. Soc. **94** (1960), 295–309.

Let $\mu \in \mathscr{P}(G)$, the semigroup of probability measures on a compact group G. Let K be the smallest closed subgroup of G that contains $S(\mu)$, the support of μ. Improving a result of Kawada and Ito [Proc. Phys.-Math. Soc. Japan (3) **22** (1940), 977–998; MR **2**, 223] the author's main theorem (3.3.5) asserts that $\mu^n \to \lambda$ (weak*) if and only if λ = Haar measure of K and $S(\mu)$ is not contained in any coset of any proper closed normal subgroup of K. The proof is Peter-Weyl-theoretic. The author also discusses (left or right) infinite products of elements from $\mathscr{P}(G)$, all such being identified when G is finite. The paper concludes with a pair of counter-examples; the first corrects a minor slip in the paper of Kawada and Ito [op. cit.], while, according to a letter from the author, the second stemmed from a misreading of Vorob'ev, Mat. Sb. (N.S.) **34** (**76**) (1954), 89–126 [MR **15**, 882].

J. G. Wendel (Ann Arbor, Mich.)

5693:

Kostant, Bertram. The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group. Amer. J. Math. **81** (1959), 973–1032.

It is impossible to do justice in a brief review to this paper which contains a wealth of results dealing with the root structure of Lie groups, the Coxeter-Killing transformation, the reduction of the restriction of the adjoint representation to a 3-dimensional subgroup (3-d.s.) and many related topics.

From a well-known theorem of H. Hopf [Ann. of Math. (2) **42** (1941), 22–52; MR **3**, 61] it follows that the homology ring of a simple Lie group G of rank n is generated by n minimal elements of odd dimension $2m_j+1$. The integers m_j have been called the exponents of the group. They determine the degrees, m_j+1, of the minimal symmetric invariants of the group, and they also determine, and are determined by, the eigenvalues of γ, the product of n simple reflections which generate the Killing (or Weyl) group. The author proposes, with reason, to call γ the Coxeter-Killing transformation. If, following Steinberg, we define the height of a positive root to be the sum of its coefficients with respect to a basis consisting of simple roots, we may formulate the following theorem: Let the number of positive roots of height $1, 2, \cdots, h-1$ be $p_1, p_2, \cdots, p_{h-1}$, respectively; then $p_1 \geqq p_2 \geqq \cdots \geqq p_{h-1}$, and the partition conjugate to that determined by the p's consists of the exponents m_j. This result was stated by Robert Steinberg [Trans. Amer. Math. Soc. **91** (1959), 493–504; MR **21** #5160], who proved it by verification. It was noticed, independently, by Arnold Shapiro who communicated it privately to the author.

The main purpose of the present paper is to prove the above theorem, which is remarkable because of the ease with which it permits the homology ring to be obtained from the root diagram. The proof is effected by establishing a relation (theorem 8.4) between the Steinberg-Shapiro procedure and the eigenvalues of γ and then by appealing to a theorem of the reviewer [Canad. J. Math. **10** (1958), 349–356; MR **21** #4990], the proof of which is improved in an important respect. In the proof of his central theorem, the author makes much use of the idea of a principal 3-d.s. (due independently to Dynkin and Siebenthal). He shows that the latter is characterized by the following property: the number of irreducible representations obtained when the adjoint representation of G is restricted to a 3-d.s., A, is $\geqq n$, with equality if and only if A is principal. Another important result characterizes the Coxeter-Killing transformations: the order of any regular element g of G is $\geqq$ the order, h, of γ, with equality if and only if g generates a Coxeter-Killing transformation. Further, the set of all regular elements of G of order h forms a conjugate class. Other results pertain to nilpotent elements and Cartan subgroups.

A. J. Coleman (Kingston, Ont.)

5694:

Гельфанд, И. М.; Минлос, Р. А.; Шапиро, З. Я. [Gel'fand, I. M.; Minlos, R. A.; Šapiro, Z. Ya.]. ★Представления группы вращении и группы Лоренца, их применения [Representations of the rotation group and of the Lorentz group, and their applications]. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 368 pp. 13.70 r.

Some years ago two of the authors [Gel'fand and Šapiro, Uspehi Mat. Nauk **7** (1952), no. 1 (47), 3–117; MR **13**, 911] published a detailed elementary account of the theory of the representations of the rotation group and its applications to the study of differential equations with rotational symmetry. According to the introduction, part I of the present work is a reprint of that article with about twenty pages of supplementary material added at the end. Part II treats the Lorentz group and its representations in much the same spirit. It was planned in detail by all three authors but actually written by Minlos. The sections dealing with Lorentz invariant equations are based upon articles by Gel'fand and Yaglom [Ž. Èksper. Teoret. Fiz. **18** (1948), 703–733, 1096–1104; MR **10**, 583, 584] and according to the authors may be considered as a more detailed and complete account of the material contained in these works. The supplementary material added to part I consists of an alternative treatment of one topic and of extensions of certain earlier discussions to more general cases.

G. W. Mackey (Cambridge, Mass.)

5695:

Lomont, J. S. Decomposition of direct products of representations of the inhomogeneous Lorentz group. J. Mathematical Phys. **1** (1960), 237–243.

The author gives a discussion of the decomposition into a continuous direct sum of the Kronecker products of certain irreducible unitary representations of the connected component of the inhomogeneous Lorentz group of four dimensions. The (single or double-valued) representations included are those corresponding to real (nonnegative) rest mass with an integer and half-integer spin. The result is that the Kronecker product of any two representations of the aforementioned class is a continuous direct sum of representations of the same class,

each of them occurring with a certain multiplicity, which is completely described. In particular, if the two factors belong to positive rest masses, then so do the irreducible components. The decomposition of the restriction of the product of two identical representation into the subspace of symmetric tensors is also discussed. The derivation of these results is sometimes rather heuristic and no attempt is made to determine the nature of the weight function of the decomposition. *Lajos Pukanszky* (College Park, Md.)

5696:

Bogaevskii, A. N. Computation of zonal spherical functions. Dokl. Akad. Nauk SSSR **129** (1959), 484–487. (Russian)

The author considers a homogeneous symmetric space $M=\mathrm{SU}(n)/\mathrm{SO}(n)$, where $\mathrm{SU}(n)$ is a unitary unimodular group of order n and $\mathrm{SO}(n)$ an orthogonal subgroup. A vector ξ is rotated in this space from a fixed initial position; the "cosine of the angle of rotation", defined as the scalar product of the initial and final values of this vector, is a zonal spherical harmonic for the space M. In this paper, the value of this harmonic is computed in terms of the elements of a suitable basis. *F. M. Arscott* (London)

5697:

Block, Richard. On Lie algebras of classical type. Proc. Amer. Math. Soc. **11** (1960), 377–379.

The author weakens the hypotheses made by the reviewer [Mem. Amer. Math. Soc. No. 19 (1956); MR **17**, 1108] in the classification of simple Lie algebras with nondegenerate trace form. It is proved that the reviewer's results still hold if one assumes only that the Lie algebra in question is simple with an ordinary representation with nondegenerate trace form, rather than being restricted with a restricted representation having nondegenerate trace form. The procedure is first to prove the algebra restricted, then that a Cartan subalgebra acts diagonally in the adjoint representation, and finally that non-zero roots are non-isotropic with respect to the form. It appears to be indicated in the proofs that by changing the representing matrices for a basis for the Lie algebra by addition of scalar matrices of trace zero a restricted representation is obtained; in any case, the theory is carried to a point from which the reviewer's results can be used. *G. B. Seligman* (New Haven, Conn.)

5698:

Ree, Rimhak. Generalized Lie elements. Canad. J. Math. **12** (1960), 493–502.

Let K be a field of characteristic zero. Given an abelian semigroup M (written additively), a bi-character of M in K is a mapping $\chi: M\times M\to K$ which is a homomorphism in each argument (regarding K as multiplicative semigroup); if, further $\chi(\rho,\sigma)\chi(\sigma,\rho)=1$ for all $\rho,\sigma\in M$ (and hence $\chi(\rho,\rho)=\pm 1$), χ is called skew-symmetric. Now let L be an algebra (not necessarily associative) over K, graded by a semigroup M, i.e., $L=\sum L_\rho$, $L_\rho L_\sigma\subseteq L_{\rho+\sigma}$, and χ a skew-symmetric bi-character of M in K. Then L is called a generalized Lie algebra of type χ or simply a χ-algebra if for any $f\in L_\rho$, $g\in L_\sigma$ and $h\in L$, $[f,g]+\chi(\rho,\sigma)[g,f]=0$, $[f,[g,h]]-\chi(\rho,\sigma)[g,[f,h]]=[[f,g],h]$. In particular, if A is an associative algebra graded by M, a χ-algebra may be defined on A by putting $[a,b]=ab-\chi(\rho,\sigma)ba$ $(a\in A_\rho,\ b\in A_\sigma)$. For any χ-algebra one can define a universal associative envelope as in the case of ordinary Lie algebras $(\chi=1)$. The author considers χ-algebras where M is a free abelian semigroup of finite rank m, generated by $\rho_1,\cdots,\rho_m$ say, and $L_{\rho_1},\cdots,L_{\rho_m}$ are each of dimension 1 and together generate L. For this M and a given χ, the notion of a free χ-algebra is defined and a basis for its universal associative envelope obtained. The subspace of terms of weight n is shown to have dimension $\psi^*(n)$ given by

$$\frac{1}{n}\sum_{d|n}\mu(d)(p+q)^{n/d}+\frac{1}{n}\sum_{2d|n}\mu(d)\{(p+q)^{n/2d}-(p-q)^{n/2d}\},$$

where p is the number of indices i with $\chi(\rho_i,\rho_i)=1$ and q the number of indices j with $\chi(\rho_j,\rho_j)=-1$ (note that the second sum is empty for odd n). The author also generalizes his algebra of shuffles [Ann. of Math. (2) **68** (1958), 210–220; MR **20** #6447] so as to yield the analogue of Friedrichs' criterion and of the Dynkin-Specht-Wever formula. *P. M. Cohn* (Manchester)

FUNCTIONS OF REAL VARIABLES

See also 5563, 5685, 5838, 5882.

5699:

Hyslop, James M. ★Real variable. University Mathematical Texts. Oliver and Boyd, Edinburgh-London; Interscience Publishers Inc., New York; 1960. viii+136 pp. $1.95.

From the introduction: "This book includes material which is taken for granted in such texts as Integration and Infinite Series, and may therefore be regarded as the foundation on which each of these rests. The book ranges over such elementary topics as bounds of sets and of functions, the theory of limits, continuity and differentiation, and the properties of the simple functions of analysis."

5700:

Kuipers, L. ★Leerboek der analyse [Textbook of analysis]. P. Noordhoff N. V., Groningen, 1960. vii+176 pp. Ing: f 15.50. Geb: f. 17.50.

This is the sort of freshman analysis course that has been fairly standard for mathematics and physics students in the Netherlands for quite some time; the reviewer is thinking for example of Kloosterman's course of the 1930's at Leiden. The book is essentially a rigorous calculus of functions of one variable, and includes the theory of infinite series. It is of some interest to look at the topics and their order. (I) The real numbers (informal discussion); (II) Sequences and limits (techniques; least upper bound axiom, Bolzano-Weierstrass theorem, Cauchy's principle); (III) Functions and limits (the standard theorems for continuous functions are proved); (IV) Differential calculus (mean value theorems, l'Hospital's rule, Taylor's formula, extreme values); (V) Integral calculus (Riemann integrability of bounded functions; rather general form of fundamental theorem, integration by parts and substitution, three mean value theorems,

improper integrals); (VI) Transcendental functions (the reviewer regrets that integral definitions have replaced the series definitions which he remembers, thus first logarithm and arc sine, then exponential and trigonometric functions; some technique of integration is included here); (VII) Series (including power series); (VIII) Uniform convergence (Weierstrass' test, integration and differentiation, Abel's theorem). If this is the direction that we want to take in this country it is clear that we still have some way to go. However, the reviewer does not advocate that we ban all applications from our beginning analysis course: the present book does not even mention the word area! It should also be noted that our newer calculus books have a better definition of function than this book's old-fashioned "y is a function of x if ...".

J. Korevaar (Madison, Wis.)

5701:

Ostrowski, A. ★Vorlesungen über Differential- und Integralrechnung. Bd. 1: Funktionen einer Variablen. 2te, neubearbeitete Aufl. Mathematische Reihe, Bd. 4. Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften. Birkhäuser Verlag, Basel-Stuttgart, 1960. 330 pp. Fr./DM 35.00.

The present edition of this book seeks to take into account various suggestions that have been made. For those students whose study of the calculus would be limited to the first volume, sections on the elements of infinite series and the elementary differential geometry of curves in the plane and euclidean 3-space have been added. The problems of the first edition have been removed and will appear together with their solutions in a special problem book. The Dedekind axiom has been replaced by the following separation axiom: If A and B are subsets of the real number system satisfying $a \leqq b$ for $(a, b) \in A \times B$, then there exists a real number s satisfying $a \leqq s \leqq b$, $(a, b) \in A \times B$. The notions of a linear operator and of a Lipschitz condition make their appearance in this edition. Sections devoted to inequalities (Jensen, Hölder, Minkowski, Bernoulli, arithmetic and geometric means) have been added.

M. H. Heins (Urbana, Ill.)

5702:

Hancock, Harris. ★Theory of maxima and minima. Dover Publications, Inc., New York, 1960. xiv+193 pp. $1.50.

Unaltered reproduction of the first edition (Ginn, 1917).

5703:

Turowicz, A. B. Sur les dérivées d'ordre supérieur d'une fonction inverse. Colloq. Math. 7 (1959), 83–87.

$x=g(y)$ is the function inverse to $y=f(x)$. The author makes use of a formula on implicit differentiation by H. A. Rothe [see E. Netto, *Lehrbuch der Kombinatorik*, Teubner, Leipzig, 1901; pp. 239–242] to obtain the formula, under the conditions that $y' \neq 0$,

$$\frac{d^n g}{dy^n} = \sum_{z_n} \frac{(2n-2-i_1)!(-1)^{n-1+i_1}}{(y')^{2n-1} \prod_{k=2}^n i_k!} \prod_{k=1}^n \left(\frac{y^{(k)}}{k!}\right)^{i_k},$$

where $z_1=0$, and for $n \geqq 2$, z_n is the set of finite sequences of n terms, $\sum_{k=2}^n (k-1)i_k = n-1$, $i_1 = n-1-\sum_{k=2}^n i_k$. For example, z_2 is the set $(0, 1)$, z_3 is the set $\{(1, 0, 1), (0, 2, 0)\}$. The formula holds for $n=1$ if all i_k are taken to be zero.

R. L. Jeffery (Wolfville, N.S.)

5704:

Doležal, Vladimír. Über den zweiten Mittelwertsatz der Integralrechnung. Časopis Pěst. Mat. 85 (1960), 84–86. (Czech. Russian and German summaries)

The author establishes the following converse of the second mean value theorem of the integral calculus. If $g \in L(a, b)$ $(-\infty < a < b < \infty)$, and if for each bounded measurable function f there exists a $\xi \in \langle a, b \rangle$ such that $\int_a^b f(x)g(x)dx = g(a) \int_a^\xi f(x)dx + g(b) \int_\xi^b f(x)dx$, then g is monotone in $\langle a, b \rangle - N$, where $N \subset \langle a, b \rangle$ is a set of measure zero.

J. F. Heyda (Cincinnati, Ohio)

5705:

Fast, H.; Urbanik, K. A characterization of step functions. Colloq. Math. 7 (1959/60), 251–254.

Let f be a Lebesgue measurable function defined on an interval I. For ε, $h > 0$, define

$$A(\varepsilon, h) = \{x : |f(x+h) - f(x)| > \varepsilon h,\ x \in I,\ x+h \in I\}.$$

The index of variability of f on I is defined by $a_I(f) = \lim_{\varepsilon \to 0} \liminf_{h \to 0+} |A(\varepsilon, h)|/h$, where $|A|$ denotes the Lebesgue measure of A. The authors prove that the values of $a_I(f)$ are either non-negative integers or $+\infty$, and that $a_I(f) = n$ $(n = 0, 1, \cdots)$ if and only if f is equivalent to a step-function with n jumps.

N. J. Fine (Philadelphia, Pa.)

5706:

Šmidov, F. I. On asymptotic differentiability of functions of two real variables. Uspehi Mat. Nauk 14 (1959), no. 4 (88), 213–216. (Russian)

The author considers measurable functions $f(x, y)$ defined on a bounded set E in the (x, y)-plane, and obtains (theorem 1) certain rather complicated necessary and sufficient conditions for f, if finite almost everywhere on E, to have an approximate total differential [cf. S. Saks, *Theory of the integral*, G. E. Stechert, New York, 1937; Chap. IX, § 12] almost everywhere on E. His conditions require that at almost every point (x_0, y_0) in E (i) f should fulfil a generalization of condition (D) [cf. loc. cit., Chap. IX, § 9], and (ii) some ray issuing from $(x_0, y_0, f(x_0, y_0))$ should not be a "generalized" intermediate half-tangent [cf. loc. cit., Chap. IX, § 2] to the surface $z = f(x, y)$, $(x, y) \in E$. He also shows (theorems 2 and 3) that if, further, f is finite throughout E, and P is the set of points on the surface at which it has an approximate tangent plane parallel to a fixed line h, then the orthogonal projection of P onto a plane perpendicular to h has zero plane measure.

H. P. Mulholland (Exeter)

MEASURE AND INTEGRATION

See also 5950.

5707:

Stromberg, Karl. A note on the convolution of regular measures. Math. Scand. 7 (1959), 347–352.

Let μ and ν be two finite Borel measures on the locally compact Hausdorff group G. Let $\mu * \nu$ be the unique regular Borel measure on G such that

$$\int_G f d(\mu * \nu) = \int_G \int_G f(xy)\, d\mu(x)\, d\nu(y)$$

for all continuous functions f on G which vanish at infinity, and define

$$(\mu\cdot\nu)(E) = \int_G \mu(Ex^{-1})\,d\nu(x)$$

for every Borel set E in G. It is a "folk theorem" that $\mu*\nu=\mu\cdot\nu$ if both μ and ν are regular measures. The author supplies a simple proof. *W. Rudin* (Madison, Wis.)

5708:

Mahowald, Mark. On relatively invariant measures. Canad. J. Math. **12** (1960), 367–373.

Let (G, S, μ) be a measure space such that G is a group and S is a σ-ring of subsets of G. It is supposed that S is left invariant ($xE\in S$ if $E\in S$ and $x\in G$), and μ is not identically equal to zero and is relatively invariant ($\mu(xE)=\sigma(x)\mu(E)$ for $E\in S$ and $x\in G$). If G is locally compact and μ is a Baire measure then the function σ is continuous [P. Halmos, *Measure theory*, Van Nostrand, New York, 1950; MR **11**, 504; p. 265]. In case G is without a topology, the present author proves that if the transformation $(x, y)\to(x, xy)$ from $G\times G$ to $G\times G$ is a measurability preserving transformation (carries $S\times S$ onto $S\times S$), then σ is S-measurable ($E\cap\sigma^{-1}(M)\cap N(\sigma)\in S$ for $E\in S$, M a Borel set of the real line and $N(\sigma)=\{x:\sigma(x)\neq 0\}$).

N. Dinculeanu (Bucharest)

5709:

Chiffi, Antonio. Sulla formula di Green nel piano. Ricerche Mat. **9** (1960), 3–19.

The version of Green's theorem proved for rectangles by Cafiero [Ricerche Mat. **2** (1953), 91–103; MR **15**, 411] is extended to simply connected regions A in the plane with boundary of finite length, under the following additional restriction on the coefficients of the form $Fdx+Gdy$ in the theorem: given $\varepsilon>0$ there exists a set I whose projection on each of the coordinate axes has measure less than ε, such that the restriction of F and G to $A-I$ is continuous.

W. H. Fleming (Providence, R.I.)

5710:

Fleming, Wendell H.; Rishel, Raymond. An integral formula for total gradient variation. Arch. Math. **11** (1960), 218–222.

Let $G\subset E_n$ be an open set, D_G the space of the infinitely differentiable functions whose supports are compact subsets of G, D_G' the Schwartz space of distributions on G, D_G^n and $D_G'^n$ the product spaces of n factors equal to D_G and D_G' respectively. If $\varphi=(\varphi_1,\cdots,\varphi_n)\in D_G^n$, $T=(T_1,\cdots,T_n)\in D_G'^n$, let $\|\varphi\|=\sup|\varphi(x)|$ for all $x\in G$, where $|\varphi|$ is the Euclidean norm, $T(\varphi)=T_1(\varphi_1)+\cdots+T_n(\varphi_n)$, and $\|T\|=\sup|T(\varphi)|$ for all $\|\varphi\|\leq 1$. If $f\in D_G'$ let grad f be the element of $D_G'^n$ defined by $(\partial f/\partial x_1,\cdots,\partial f/\partial x_n)$. If E is a measurable set in G, let χ_E denote the characteristic function of E. For any real-valued locally integrable function f in G, the identity holds

$$\|\text{grad } f\| = \int_{-\infty}^{+\infty}\|\text{grad }\chi_{E_t}\|\,dt,$$

where $E_t=[x\in G\,|\,f(x)>t]$. This relation extends, by methods of the theory of distributions, an elementary identity which was first extended by H. Federer to Lipschitzian functions f [Trans. Amer. Math. Soc. **93** (1959), 418–491; MR **22** #961]. The present definition of $\|\text{grad }\chi_{E_t}\|$ extends one of R. Caccioppoli [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12** (1952), 3–11, 137–146; MR **13**, 830, 925] and E. De Giorgi [Ann. Mat. Pura Appl. (4) **36** (1954), 191–213; MR **15**, 945].

L. Cesari (Ann Arbor, Mich.)

FUNCTIONS OF COMPLEX VARIABLES

See also 5549, 5813, 5839, 5985, 5986, 5987.

5711:

★Исследования по современным проблемам теории функций комплексного переменного [Investigations in contemporary problems of the theory of functions of a complex variable]. Collection of articles edited by A. I. Markuševič. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. 544 pp. 18.05 r.

48 papers, based on talks given at the Third All-Union Conference on the Theory of Functions of a Complex Variable, Moscow, 28 May to 2 June, 1957. Each paper will receive an independent review in MR.

5712:

Tsuji, M. ★Potential theory in modern function theory. Maruzen Co., Ltd., Tokyo, 1959. 590 pp. $12.50.

Over a period of several decades the author has contributed to the theory of functions an impressive array of notes and papers, ranging from remarks on classical theorems to ingenious proofs of important new results. The present book serves first of all as a synthesis of the extensive literature of the author, some of which is virtually inaccessible, and second as a sketch of the background material needed to read the author's papers. The table of contents shows the broad range of the author's interests: Chapter 1, The Dirichlet problem; Chapter 2, Subharmonic functions; Chapter 3, Topics in potential theory; Chapter 4, The Poisson integral; Chapter 5, Nevanlinna's theory of meromorphic functions; Chapter 6, Ahlfors' theory of covering surfaces; Chapter 7, Directions of Borel; Chapter 8, Cluster sets of meromorphic functions; Chapter 9, Conformal mapping; Chapter 10, Riemann surfaces; Chapter 11, Fuchsian groups. There is a short appendix on the Birkhoff ergodic theorem.

A. J. Lohwater (Houston, Tex.)

5713:

Nehari, Zeev. On the derivative of an analytic function. Amer. Math. Monthly **67** (1960), 446–448.

A proof is given that existence and boundedness of the derivative of a complex function $f(z)$ implies continuity of $f'(z)$. Green's theorem is avoided but Fubini's theorem is used to derive a mean value theorem and hence a form of the Poisson integral formula.

W. Kaplan (Ann Arbor, Mich.)

5714:

Meier, Kurt. Eine hinreichende Bedingung für die Regularität einer komplexen Funktion. Comment. Math. Helv. **34** (1960), 67–70.

The author succeeds in weakening considerably the hypotheses of the classical Looman-Menchoff theorem. His result is as follows. Let $f(z)$ $(z=x+iy)$ be continuous in a region G. If there exists a sequence $\{r_n\}$ of positive numbers tending to zero such that, for every $z \in G$, the four difference quotients $(f(z+r_n)-f(z))/r_n$, $(f(z+ir_n)-f(z))/ir_n$, $(f(z-r_n)-f(z))/(-r_n)$, $(f(z-ir_n)-f(z))/(-ir_n)$ all approach the same finite limit as $n \to \infty$, then $f(z)$ is holomorphic in G. *W. Seidel* (Notre Dame, Ind.)

5715:

Nicolescu, Lilly-Jeanne. La dérivée aréolaire des fonctions d'une variable complexe à valeurs dans un espace de Banach. Com. Acad. R. P. Romîne **9** (1959), 1007–1012. (Romanian. Russian and French summaries)

The so-called "areolar derivative" as defined by Pompeiu and N. Teodoresco is extended from the complex-valued case to the Banach-space-valued case. The vector-valued extensions of some results of N. Teodoresco are also derived. *S. Zaidman* (Bucharest)

5716:

Bers, Lipman. Quasiconformal mappings and Teichmüller's theorem. Analytic functions, pp. 89–119. Princeton Univ. Press, Princeton, N.J., 1960.

There has been an unfortunate gap of three years between the oral and printed form of this communication. Even though the author has managed to sneak in some references to later work, it is clear that the paper would have been much more widely read if publication had been prompt. It contains (1) a very clear account of the relationship between Beltrami differentials and quasiconformal mappings; (2) a precise and clear proof of Teichmüller's theorem on extremal quasiconformal mappings. The proof is based on a continuity method, and as such it is very closely related to Teichmüller's own proof. Needless to say, all traces of obscurity have been removed. *L. Ahlfors* (Cambridge, Mass.)

5717:

Chen, Kien-kwong. On the Hölder exponent of Q-mapping. Sci. Record (N.S.) **3** (1959), 393–399.

It is known that a Q-quasiconformal function $w(z)=u(x,y)+iv(x,y)$, i.e., a continuous L_2-differentiable solution of the differential inequality $|\partial w/\partial \bar{z}| \leq k|\partial w/\partial z|$, with $0 \leq k=(Q-1)/(Q+1)<1$, satisfies a Hölder condition with exponent $\alpha=1/Q$. The author shows that if w has a differential everywhere and $|w_x|^2+|w_y|^2$ is subharmonic, then the same is true for $\alpha=2/Q$, provided that $Q>2$. If, in addition, w is a homeomorphism of $|z| \leq 1$ onto itself which leaves $z=0$ fixed, then the Hölder constant can be estimated in terms of Q. *L. Bers* (New York)

5718:

Portmann, Walter O. Hausdorff-analytic functions of matrices. Proc. Amer. Math. Soc. **11** (1960), 97–101.

The author extends the concept of Hausdorff-analytic functions of a hypercomplex system [F. Hausdorff, Leipziger Berichte **52** (1900), 43–61] to a set of square matrices with distinct eigenvalues. He proves that if a matrix Z has distinct eigenvalues at Z_0, then Z is Hausdorff-analytic in some neighborhood N of Z_0. For each such Z in N there exists a Q whose elements are analytic functions of the elements of Z, and such that QZQ^{-1} is a diagonal matrix. The proofs are by means of the Frobenius covariants of Z. *J. A. Ward* (Washington, D.C.)

5719:

Portmann, Walter O. A sufficient condition for a matric function to be a primary matric function. Proc. Amer. Math. Soc. **11** (1960), 102–106.

Let $f(z)$ be a scalar function analytic at the eigenvalues of Z in M (the algebra of n by n matrices over the complex field). Define

$$f(Z) = (2\pi i)^{-1}\int_C f(\lambda)(\lambda I-Z)^{-1}\,d\lambda,$$

where C is a set of admissible closed paths around the eigenvalues of Z, and the integration is an elementwise scalar integration. Such a function $f(Z)$ is called (by Frobenius) a primary matrix function. The author gives a set of necessary and sufficient conditions on a matrix function $G(Z)$ in order that there exist an $f(z)$ such that $G(Z)=f(Z)$. *J. A. Ward* (Washington, D.C.)

5720:

Callahan, Francis P., Jr. An extremal problem for polynomials. Proc. Amer. Math. Soc. **10** (1959), 754–755.

Let $f_n(z)$ be a polynomial of order n at most, $f_n(1)=0$, $|f_n(z)| \leq 1$ for $|z| \leq 1$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} |f_n(e^{i\theta})|^2\,d\theta \leq \frac{n}{n+1}.$$

The polynomials for which this extremal value is attained, are determined. *G. Freud* (Budapest)

5721:

Roux, Delfina. Una dimostrazione del Teorema fondamentale dell'Algebra. Boll. Un. Mat. Ital. (3) **14** (1959), 563–567. (English summary)

In the paper the author gives another proof of the fundamental theorem of algebra. If $P(z)$ is a polynomial the proof follows from a study of the coefficients in the power series expansion of $1/P(z)$. *V. F. Cowling* (Lexington, Ky.)

5722:

Pommiez, Michel. Sur les restes successifs des séries de Taylor. C. R. Acad. Sci. Paris **250** (1960), 2669–2671.

Let $f(z)=\sum_0^\infty a_p z^p$ be regular in $|z|<1$, and set $f_n(z)=\sum_0^\infty a_{n+p}z^p=z^{-n}[f(z)-a_0-a_1z-\cdots-a_{n-1}z^{n-1}]$. Using a completeness theorem of Boas [Trans. Amer. Math. Soc. **48** (1940), 467–487; MR **2**, 80], the author proves that if $|\alpha|<1$ and $|z_n-\alpha| \leq r < \frac{1}{2}(1-|\alpha|)$, $n=1, \cdots, \infty$, there exists a sequence of polynomials $P_n(z)$ of degree n and depending only on the z_n such that $f(z)=\sum_{n=0}^\infty P_n(z)f_n(z_n)$. In particular if $f_n(z_n)=0$ for all sufficiently large n, $f(z)$ is a polynomial. If $\alpha=0$ the condition $|z_n| \leq r < \frac{1}{2}$ cannot be replaced by $|z_n|<.5881$ in these hypotheses. Again if $|\alpha_n| \leq r$, $|\beta_n| \leq r$, $n=1,2,\cdots$, where $r<1-2^{-1/2}$, the author establishes similarly an expansion

$$f(z) = \sum_{n=1}^\infty c_n(z)\frac{f_{n-1}(\alpha_n)-f_{n-1}(\beta_n)}{\alpha_n-\beta_n},$$

where the $c_n(z)$ are polynomials of degree n. Hence unless $f(z)$ is a polynomial, infinitely many of the functions $f_n(z)$ are univalent in $|z|<r$. Here the upper bound $1-2^{-1/2}$ for r cannot be replaced by any number greater than $2^{1/2}-1$, as is shown by the functions $f(z)=(1+z)/(1+z^2)$ for which $f_n(z)=(\mp 1\mp z)/(1+z^2)$, so that $f_n'(z)=0$ if $z=2^{1/2}-1$, or $1-2^{1/2}$.

The author also shows by elementary methods that if $f(z)$ is an integral function of finite order and has a finite Picard value, $f_n(z)$ cannot have any finite Picard value except possibly zero. {The reviewer remarks that if $f(z)-a$ and $f_n(z)-b$ have only a finite number of zeros, then

$$\phi(z) = \frac{f(z)-a}{a_0+a_1z+\cdots+a_{n-1}z^{n-1}+bz^n-a}$$

has only a finite number of zeros, ones and poles so that the above result is a direct consequence of Picard's theorem even for general transcendental integral functions $f(z)$. The exceptional case $b=0$ can occur only if $a=a_0$ and $a_1=a_2=\cdots=a_{n-1}=0$, so that $f_n(z)\equiv[f(z)-a]z^{-n}$.}

W. K. Hayman (London)

5723:

Resegotti, Enrico. Sulla classe limite della funzione e^{1/z^n} per $z\to 0$ lungo un arco regolare. Boll. Un. Mat. Ital. (3) **15** (1960), 43–46. (English summary)

The author determines the possible limiting values of $w=\exp(z^{-n})$ when z tends to zero along a curve with equation

$$\varphi = \alpha_0+\alpha_1\rho+\alpha_2\rho^2+\cdots.$$

These limiting values are zero or infinity except for special values of the coefficients α_i.

L. M. Graves (Chicago, Ill.)

5724:

Berg, Lothar. Komplexe Kurvenintegrale über nicht-rektifizierbare Kurven. Math. Nachr. **20** (1959), 259–264.

Let G be a simply-connected region in the complex plane on which f is analytic and let $C: z=z(t)$, $\alpha\leqq t\leqq\beta$, be a parametric curve in G. Let $\alpha=t_0<t_1<\cdots<t_n=\beta$, $z_\nu=z(t_\nu)$, $\zeta_\nu=z(\tau_\nu)$ where $t_{\nu-1}\leqq\tau_\nu\leqq t_\nu$. Suppose that $\lim_{n\to\infty}\sum_{\nu=1}^n f(\zeta_\nu)(z_\nu-z_{\nu-1})$ exists, where

$$\lim_{n\to\infty}\max\,(t_\nu-t_{\nu-1}) = 0.$$

The limit is denoted by $\int_C f(z)dz$ even if C is not rectifiable. A necessary and sufficient condition for the limit to exist is

$$\lim_{n\to\infty}\left(\int_{z_{\nu-1}}^{\zeta_\nu} f'(\zeta)(z_{\nu-1}-\zeta)\,d\zeta+\int_{\zeta_\nu}^{z_\nu} f'(\zeta)(z_\nu-\zeta)\,d\zeta\right)\to c.$$

If $c=0$ for a class of curves joining $z(\alpha)$ to $z(\beta)$, then the integral is independent of the path in the class. The collection of curves each having a parametric representation satisfying a Hölder condition with exponent greater than $\frac{1}{2}$ is such a class. If $f(z)=z$ then the integral is independent of the path, whenever it exists.

E. Silverman (Lafayette, Ind.)

5725:

Iacob, Caius. Sur les solutions à singularités données des problèmes de Riemann et de Hilbert. Acad. R. P. Romîne. Stud. Cerc. Mat. **10** (1959), 255–272. (Romanian. Russian and French summaries)

Riemann's problem with singularities is the following. In the complex plane, D_1 represents a simply connected domain bounded by a smooth closed curve C; D_2 is the exterior of C. For $j=1, 2$, $F_j(z)$ is a function analytic in the entire plane except for isolated singularities in D_j. To find a function $f(z)$ piecewise analytic such that $f(z)=f_j(z)$ for $z\in D_j$, where $f_j(z)$ is a given function continuous on C and satisfying: the difference $h_j(z)-f_j(z)-F_j(z)$ is continuous in $D_j\cup C$ and holomorphic in D_j; on C, one has $f_1(\xi)=G(\xi)f_2(\xi)+L(\xi)$, $\xi\in C$, where G and L are given Hölder functions, $G(\xi)\neq 0$. The author notes that this problem can be solved by reducing it to the Riemann problem without singularities, whose solution is known. However, he develops a direct method more useful for applications. The solution to the problem always exists if $\chi\geqq 0$ where $2\pi i\chi$ is the variation of $\log G(\xi)$ around C. If $\chi<0$, $\chi=-q$, a certain function must have a zero of order q at ∞.

Hilbert's interior problem with singularities is as follows: Given $F(z)=A(z)+iB(z)$, analytic in the entire plane, with isolated singularities in D_1 and holomorphic in $D_2\cup C$. To find $f(z)=u+iv$ such that: (a) $f(z)-F(z)$ is continuous in $D_1\cup C$ and holomorphic in D_1; (b) on C one has $a(\xi)u(\xi)+b(\xi)v(\xi)=c(\xi)$, $\xi\in C$, where a, b, c are Hölder functions, $a^2+b^2=1$. The author gives two solutions of this problem, one by integral equations, the other by reducing it to Riemann's problem. There are three applications.

E. R. Lorch (New York)

5726:

Baganas, Nicolas. Un nouveau critère de normalité d'une famille de fonctions algébroïdes. C. R. Acad. Sci. Paris **250** (1960), 1424–1425.

The author announces the theorem that the family of algebroidal functions of n branches, which are defined in some domain D and which do not assume at the same point $z\in D$ any of the values of a given algebroidal function having at least $2n+1$ distinct branches, is normal.

A. J. Lohwater (Houston, Tex.)

5727:

Tsuji, Masatsugu. Huber's theorem on analytical mappings of a ring domain in a ring domain. Comment. Math. Univ. St. Paul **8** (1960), 41–43.

A new proof is given of a theorem due to H. Huber [Compositio Math. **9** (1951), 161–168; MR **13**, 337].

W. Seidel (Notre Dame, Ind.)

5728:

Schnyder, Adolf Theophil. Über ein vollständiges System konformer Invarianten von dreifach-zusammenhängenden Gebieten. Comment. Math. Helv. **34** (1960), 85–98.

For a triply connected region G, let $\lambda[\Gamma_i, G]$ be the extremal length of the family of curves which separate a contour Γ_i $(i=1, 2, 3)$ from the others, and set $\mu_i(G)=\lambda^{-1}[\Gamma_i, G]$. Two regions are conformally equivalent if and only if they have the same $\mu_i(G)$. These quantities satisfy the inequalities $\mu_i<\mu_j+\mu_k$ and are otherwise completely independent. By the simplicity of the relations, the result constitutes an improvement on earlier characterizations.

L. Ahlfors (Cambridge, Mass.)

5729:

Ahlfors, Lars V.; Sario, Leo. ★**Riemann surfaces.** Princeton Mathematical Series, No. 26. Princeton University Press, Princeton, N.J., 1960. xi + 382 pp. $10.00.

A glance at the vast and very comprehensive bibliography of the book under review furnishes convincing evidence that there has been considerable activity in the study of the theory of Riemann surfaces since 1945 and that the theory has many different facets. Indeed, the output has been so great that it was necessary for great selectivity to be exercised in writing what must certainly be regarded as one of the authoritative accounts of the modern theory of Riemann surfaces. The choice of the material included and its disposition in the text depend upon many factors: the critical judgment of the authors, the importance of the material, and contemporary interest.

Perhaps the spirit of the book can best be described by saying that it concentrates principally on the foundations essential for an understanding of the modern theory of Riemann surfaces and treats these meticulously. It is definitely intended that the treatise be accessible to students whose background may be limited. The student who has mastered the text will be very adequately prepared to pursue more special studies in the theory of Riemann surfaces.

It is not possible to include in a review any account of the history of the subject, which is over a century in age, but it should be pointed out that at different periods there were different preoccupations. Some of these led to classic results that became incorporated in many accounts of the subject. One topic that engaged the energies of many distinguished mathematicians was the fusion of the Riemannian and Weierstrassian points of view. We mention here the endowing of a Weierstrassian analytische Gebilde with a conformal structure, and problems such as the determination of an ordered pair of meromorphic functions on a plane region which induce all the elements of a Gebilde (and, in particular, a pair whose components are possibly subject to the requirement that their domain be of some simple nature), and the Riemann-Koebe problem of determining a meromorphic function "belonging to a concrete Riemann surface". Recent monographs on Riemann surfaces prior to that of Ahlfors and Sario have adhered to a historically orthodox approach and individually treat some aspect of the "fusion problem". The present treatise breaks with tradition on this point. There is a brief preconformal glance at the Weierstrassian Gebilde in the study of covering surfaces, but nothing is said about the "fusion question". To complain on this score is to cavil. A selection of topics was inevitable and it may well be argued that the original "fusion" problems no longer retain the dominating interest they formerly had. But it is not amiss to express the hope that the authors may include in a subsequent edition an account of the historical development of the theory of Riemann surfaces which would give the interested reader some notion of the rich topography of the subject, and also specific bibliographical indications for the study of areas not treated in the text. It would be hard to think of authors better equipped to do this.

A topic that the authors omitted regretfully was the theorem of Behnke and Stein and the subsequent extension of the Weierstrass infinite product theorem and the Mittag-Leffler theorem given by H. Florack in her thesis. [As the authors remark, an account of this material is to be found in Behnke and Sommer, *Theorie der analytischen Funktionen einer komplexen Veränderlichen*, Springer, Berlin, 1955; MR 17, 470.] However, the reader is admirably prepared to take up these matters after the study of the present book. The investigation of algebraic structures associated with a given Riemann surface (rings of analytic functions, fields of meromorphic functions), which received considerable impetus from the work of Chevalley and Kakutani and the work of Bers, and in recent times has evoked active interest, is not taken into account. But the compensations for such omissions are many. The fundamental existence theorems are treated in a throughgoing way, and honor is given to methods involving the Dirichlet integral, to operator methods that find their source in the ideas of H. A. Schwarz, and to the method of Perron using subharmonic functions. A detailed account of classification theory, which owes much to the authors, is given. It carries the subject forward to the stage of contemporary investigations. Similarly, the study of the periods and singularities of differentials on non-compact ("open") Riemann surfaces, a subject of lively interest, is given a thoroughly up-to-date account.

The book is divided into five major chapters: 1. Surface topology, 2. Riemann surfaces, 3. Harmonic functions on Riemann surfaces, 4. Classification theory, 5. Differentials on Riemann surfaces. It is noted that chapters 1, 2, 5 are the outgrowth of lectures given by Ahlfors during the past decade, that chapter 3 is based on the research of Sario on the operator method that he developed for the treatment of the fundamental existence theorems, and that the responsibility for the material of chapter 4 is shared by both authors. This remark refers to the sources of the book's material. The final version represents the reworking and polishing of about a decade shared by both authors.

Chapter 1 is concerned with surfaces, covering surfaces, simplicial homology and singular homology, compactification of surfaces, classification of polyhedra. It terminates in a proof of the theorem of Radó which states that a surface with countable base is triangulable. Thanks to this result, an essential tool for the proof of Stoilow's topological characterization of a Riemann surface is made available. The authors remark that in an otherwise self-contained presentation they take for granted the Jordan-Schoenflies theorem since its proof casts no light on the problems of primary concern.

Chapter 2 treats the definition of the basic concepts of Riemann surface theory, harmonic functions, Dirichlet integral, the solution of the Dirichlet problem by the method of Perron. A proof of the theorem of Radó which asserts that a Riemann surface has a countable base is given. It uses the idea of the proof given by R. Nevanlinna, which introduces a metric with the aid of a non-constant harmonic function.

The fundamental existence theorems are established in chapter 3 by the use of Sario's operator method, which here receives a unified general treatment. A basic application is to a proof of the celebrated theorem which asserts that a planar Riemann surface is conformally equivalent to a plane region. The chapter terminates with a generalization of the notion of logarithmic capacity and mappings of planar surfaces onto slit plane disks.

It is not possible to give more than a superficial idea of the wealth of material in the chapter on classification theory. In brief, the contents of chapter 4 are: § 1

fundamental theorems concerning degeneracy classes, § 2 positive harmonic functions, § 3 the method of extremal length, §§ 4–5 criteria, § 6 applications, § 7 plane regions, § 8 counterexamples. In this last section counterexamples are given which show that the basic inclusion relations of classification theory are strict.

The final chapter opens with an account of differential forms. The method of orthogonal projection follows. The next section is concerned with the periods and singularities of harmonic and analytic differentials. It culminates in the search of analogues of Abel's theorem and Riemann's bilinear relation for non-compact Riemann surfaces. The questions that arise here are far from settled and much remains to be done in this frontier domain of modern Riemann surface theory. The last section turns to the case of compact Riemann surfaces and establishes two of the glories of the classical theory: the Riemann-Roch theorem and the Weierstrass gap theorem.

Indications of open questions are given at various places in the text. *M. H. Heins* (Urbana, Ill.)

5730:

Gol'dberg, A. A. A class of Riemann surfaces. Mat. Sb. (N.S.) **49 (91)** (1959), 447–458. (Russian)

The author generalizes the Riemann surfaces with periodic, doubly periodic and quarter periodic ends considered by Künzi [see, e.g., Comment. Math. Helv. **30** (1956), 107–115; MR **17**, 837] by introducing surfaces with a finite number of π-ends. A π-end is obtained by extracting a sector from the graph of the Riemann surface of a doubly periodic function. An explicit quasi-conformal mapping of each surface onto the finite z-plane is given, so that each surface is of parabolic type. Simple formulas are given for distribution of values, asymptotic paths and asymptotic values of the corresponding entire function. *W. Kaplan* (Ann Arbor, Mich.)

5731:

Oikawa, Kôtaro. Sario's lemma on harmonic functions. Proc. Amer. Math. Soc. **11** (1960), 425–428.

Given a compact set K on an open Riemann surface W, there exists a positive constant $q<1$ such that

$$(1)\qquad \max_K |u| \leq q\cdot\sup_W |u|$$

for every harmonic function u which changes sign on K [L. Sario, Trans. Amer. Math. Soc. **72** (1952), 281–295; MR **13**, 735]. The author is concerned with evaluating q. If W is the disk $|z|<1$, let $d(z_1, z_2)$ be the distance between $z_1, z_2 \in K$ in the metric $ds=|dz|/(1-|z|^2)$. The supremum of $d(z_1, z_2)$ as z_1, z_2 range in K is denoted by D. The author shows that the smallest possible value of q is

$$(2)\qquad \frac{2}{\pi}\,\mathrm{Arctan}\sinh 2D,$$

where Arctan indicates the principal value of arctan. The proof is based on the following lemma suggested to the author by S. Warschawski: If $u(z)$ is harmonic in $|z|<1$ with $u(0)=0$ and $|u(z)|\leq 1$, then

$$(3)\qquad |u(z)| \leq \frac{2}{\pi}\,\mathrm{Arctan}\,\frac{1}{1-r^2}$$

in $|z|\leq r<1$.

For a hyperbolic Riemann surface W, the author defines the diameter D of K analogously in terms of the Poincaré metric on the conformal image $|z|<1$ of the universal covering surface of W. Then (2) continues to qualify as a value for q in (1) but the question remains open whether it is the smallest possible value in the case that $W \notin O_{HB}$. *L. Sario* (Los Angeles, Calif.)

5732:

Christian, Ulrich. Über die Multiplikatorensysteme zur Gruppe der ganzen Modulsubstitutionen n-ten Grades. Math. Ann. **138** (1959), 363–397.

Consider the group of analytic automorphisms of the generalized upper half-plane of degree n [Siegel, Math. Ann. **116** (1939), 617–657] of the form $T: Z \to U'ZU+S$, where S is a symmetric integral matrix, U is a unimodular integral matrix, and U' is the transpose of U. A factor of automorphy (Multiplikatorensysteme) for this group is a set of holomorphic, nowhere-vanishing functions $I(T, Z)$ associated to the transformations T of the group such that $I(T_1T_2, Z)=I(T_1, T_2Z)I(T_2, Z)$ for each pair of transformations T_1, T_2. The author gives a complete classification of these factors of automorphy for $n\geq 3$; modulo the usual notion of equivalence [Gunning, Amer. J. Math. **78** (1956), 357–382; MR **18**, 933] the only factors of automorphy are the trivial factor $I(T, Z)=1$ and, for even n, the factor $I(T, Z)=\det U$ when the transformation T is written as above. This classification involves, in addition to the more or less standard analytical techniques, a careful study of the algebraic properties of the group of unimodular matrices. *R. C. Gunning* (Princeton, N.J.)

5733:

Potapov, V. P. The multiplicative structure of J-contractive matrix functions. Amer. Math. Soc. Transl. (2) **15** (1960), 131–243.

Translation of Trudy Moskov. Mat. Obšč. **4** (1955), 125–236 [MR **17**, 958].

5734:

Ivanov, V. K. The growth characteristic of entire functions of several complex variables. Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo, pp. 301–305. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

Summary of the author's paper in Mat. Sb. (N.S.) **47 (89)** (1959), 3–16 [MR **21** #3580]. *R. P. Boas, Jr.* (Evanston, Ill.)

5735:

Bitlyan, I. F.; Gol'dberg, A. A. The Wiman-Valiron theorems for integral functions of several complex variables. Vestnik Leningrad. Univ. **14** (1959), no. 13, 27–41. (Russian. English summary)

The theorems in question, in the one-variable case, connect the maximum modulus $M(r)$, maximum term $m(r)$ {note the unconventional notation} and central index $\nu(r)$. They state that

$$M(r) \leq Cm(r)\{\log m(r)\}^{(1/2)+\varepsilon};$$

and if $|f(\zeta)|=M(r)$, $|\zeta|=r$, then $\lim \zeta f'(\zeta)/\{\nu(r)f(\zeta)\}=1$, in each case with the possible exception of a set of finite logarithmic measure. The authors present two extensions

to functions of two variables. First let $M(r_1, r_2) = \max |f(z, w)|$ for $|z| = r_1$, $|w| = r_2$; and $m(r_1, r_2) = \max |a_{nm}| r_1^n r_2^m$ for $|z| \le r_1$, $|w| \le r_2$; then $M(r_1, r_2) \le Cm(r_1, r_2)\{\log m(r_1, r_2)\}^{(3/2)+\epsilon}$ except at most for a set in the quadrant $r_1 \ge 0$, $r_2 \ge 0$ of finite logarithmic measure on each ray. Next let $\nu(r_1, r_2)$ be the largest n for which $|a_{nm}| r_1^n r_2^m = m(r_1, r_2)$ and $\mu(r_1, r_2)$ the largest m for which $|a_{\nu m} r_1^\nu r_2^m| = m(r_1, r_2)$. Then if f is not a polynomial but $a_{nm} = 0$ for $m > C_1 n^\beta + C_2$, where $0 < \beta < 1$, then

$$\lim_{r_1+r_2\to\infty} \frac{\zeta \partial f(\zeta, \omega)/\partial z + \omega \partial f(\zeta, \omega)/\partial w}{\{\nu(r_1, r_2) + \mu(r_1, r_2)\} f(\zeta, \omega)} = 1,$$

where $|\zeta| = r_1$, $|\omega| = r_2$, $|f(\zeta, \omega)| = M(r_1, r_2)$; examples show that the extra condition on a_{nm} is essential. The authors next define maximum modulus, maximum term, and central index in a different way. Write

$$f(z, w) = \sum_{k=0}^{\infty} \Big(\sum_{l=0}^{k} a_{k-l,l} z^{k-l} w^l \Big) = \sum_{k=0}^{\infty} A_k(z, w).$$

Let S_1 be a circular region centered at the origin, i.e., $(z_0 e^{i\theta}, w_0 e^{i\theta}) \in S_1$ when $(z_0, w_0) \in S_1$; let S_r be the set of (z, w) with $(z/r, w/r) \in S_1$. Then define $\mathbf{M}(r, f) = \max |f(z, w)|$ over S_r, attained for $(\tilde z, \tilde w)$ on the boundary of S_r; $\mathbf{m}_k(r, f) = \max |A_k(z, w)|$; $\mathbf{m}(r, f) = \max_k \mathbf{m}_k(r, f)$; κ the largest k for which $\mathbf{m}_k(r, f) = \mathbf{m}(r, f)$; then

$$\mathbf{m}(r, f) \le \mathbf{M}(r, f) \le C\mathbf{m}(r, f)\{\log \mathbf{m}(r, f)\}^{(1/2)+\epsilon},$$

and

$$\lim_{r\to\infty} \frac{\tilde z \partial f(\tilde z, \tilde w)/\partial z + \tilde w \partial f(\tilde z, \tilde w)/\partial w}{\kappa(r) f(\tilde z, \tilde w)} = 1,$$

in each case with the possible exception of sets of finite logarithmic measure. *R. P. Boas, Jr.* (Evanston, Ill.)

5736:

Ibragimov, I. I. Some inequalities for entire functions of exponential type. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 605–616. (Russian)

Let $f(z)$ be an entire function of exponential type σ, belonging to L^p on the real axis. The author extends previous results [see in particular his papers, Izv. Akad. Nauk Azerbaĭdžan. SSR. Ser. Fiz.-Teh. Him. Nauk **1958**, no. 2, 3–17; Dokl. Akad. Nauk SSSR **121** (1958), 415–417; MR **22** #871; **20** #6628] as follows. If $2 < p < p' \le \infty$, then

$$\|f\|_{p'} \le (p\sigma/\pi)^{(1/p)-(1/p')} \|f\|_p;$$

if $p \ge 1$,

$$\max |f(x+iy) \sin \alpha + \bar f(x+iy) \cos \alpha| \le \cdot \left\{ \frac{\sinh p\sigma y}{\pi p y} \right\}^{1/p} \|f(x)\|_p$$

for arbitrary real α; if $1 \le p \le 2$ and $1 \le p < p' \le \infty$,

$$\|f(x+iy)e^{-i\omega} + f(x-iy)e^{i\omega}\|_{p'} \le 2(\sigma/\pi)^{(1/p)-(1/p')}(\cosh^2 \sigma y - \sin^2 \omega)^{1/2} \|f(x)\|_p$$

for arbitrary real ω. The author also gives some corollaries and extensions to derivatives of f and to several variables.
R. P. Boas, Jr. (Evanston, Ill.)

5737:

Dwivedi, S. H. On entire functions of finite order. Math. Student **26** (1958), 169–172.

Let $\phi(x) > 0$ be a continuous function such that $\int_A^\infty dx/[x\phi(x)] < \infty$, and define $N(r) = A + \int_{r_0}^r [n(t)/t] dt$, where $n(r)$ is the number of zeros in $|z| \le r$ of an entire function of finite order equal to its genus. Then

$$\limsup_{r=\infty} N(r)\phi(r)/\log M(r) = \infty.$$

A. G. Azpeitia (Providence, R.I.)

5738:

Ibragimov, I. I. Extremal properties of entire functions of finite type. Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo, pp. 175–185. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

The author extends his method [Uspehi Mat. Nauk **12** (1957), no. 3 (75), 323–328; MR **19**, 737] for solving extremal problems in the class of entire functions f of order one and type σ and belonging to L_2 on the real axis (and its extension to functions of several complex variables [Dokl. Akad. Nauk SSSR **128** (1959), 1114–1117; MR **22** #103]) to the estimation of linear functionals of f in terms of the L_1-norm $\|f\|$ of f. Example: For real a and b

$$\left| \int_a^b f(x)\, dx \right| \le \sigma |b-a| \|f\|/\pi.$$

W. H. J. Fuchs (Ithaca, N.Y.)

5739:

Edrei, Albert; Shah, S. M. A conjecture of R. Nevanlinna concerning the genus of a meromorphic function. Proc. Amer. Math. Soc. **11** (1960), 319–323.

A conjecture of R. Nevanlinna [*Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Gauthier-Villars, Paris, 1929; p. 106] concerning the deficiency of a meromorphic function, its genus, and the genus of its derivative, is settled negatively by constructing an entire function f of order one and of regular growth satisfying: (1) the genus of f' is less than that of f, (2) $\sum_{a\ne\infty} \delta(a, f) = 0$, where $\delta(a, f)$ is the deficiency of a with respect to f.
M. H. Heins (Urbana, Ill.)

5740a:

Pommerenke, Christian. Über die Kapazität ebener Kontinuen. Math. Ann. **139**, 64–75 (1959).

5740b:

Pommerenke, Christian. Über die Kapazität der Summe von Kontinuen. Math. Ann. **139**, 127–132 (1959).

5740c:

Pommerenke, Christian. Über die analytische Kapazität. Arch. Math. **11** (1960), 270–277.

Let E be a plane compact set. Let cap E be the capacity or transfinite diameter of E. In the first paper the author obtains new estimates for cap E in terms of various geometrical quantities such as the width b of E, the area A of E and circumference L_0 of the convex hull of E. Most of the results are only true when E is connected. With this assumption and if cap $E = 1$, the author proves for instance that $L_0 \le 9.173$, $b \le \pi^{1/2} 3^{1/4}$. If E is symmetric about the origin then $b \le 2$, with equality for a circle. If E is

invariant under the transformation $w'=we^{2\pi i/n}$, then

$$b \geqq \frac{2\pi}{n}\frac{\Gamma(1+2/n)}{\Gamma(1+1/n)^2}\cot\frac{\pi}{n}, \quad n \text{ even},$$

(1)

$$\geqq \frac{\pi}{n}\frac{\Gamma(1+1/n)}{\Gamma(1+2/n)^2}\cot\frac{\pi}{2n}, \quad n \text{ odd},$$

with equality for the regular n-gon or the regular $2n$-gon respectively. If E contains a segment of length d, where d is the diameter of E, then $b^2+d^2 \leqq 16$, with equality for a straight line segment. For a curve of constant width we have $2 \leqq b \leqq 2.112$.

Suppose that $f(z)=z+a_0+\cdots$ is regular in $|z|>1$. Then if E_0 is the set of values not taken by $f(z)$ in $|z|>1$, the reviewer proved [J. Analyse Math. **1** (1951), 155–179; MR **13**, 545] that cap $E_0 \leqq 1$. In the opposite direction the author shows in the second paper that if E is the cluster set of $f(z)$ as $|z|\to 1$, then

(2) $$\text{cap } E \geqq 1.$$

Equality holds again if and only if $f(z)$ is univalent. Let $E_1, \cdots, E_n$ be continua. The Minkowski sum $E=E_1+E_2+\cdots+E_n$ is defined to be the set of points of the form $z=z_1+z_2+\cdots+z_n$, where $z_j \in E_j$. Using (2) the author proves that

(3) $$\text{cap } E \geqq \text{cap } E_1+\cdots+\text{cap } E_n.$$

This yields in particular cap $E^* \geqq$ cap E, where E^* consists of all points of the form $z=\frac{1}{2}(z_1-z_2)$ where z_1, z_2 are points of E. These results are used to give a new proof of (1), and of the sharp inequality of Pólya and Schiffer [C. R. Acad. Sci. Paris **248** (1959), 2837–2839; MR **21** #4247]: $L \leqq 8$ cap K, where K is a convex continuum whose circumference has length L. The author also proves that if E is a continuum of capacity 1 and $w=f(z)=z+a_0+\cdots$ maps $|z|>1$ conformally onto the exterior of E, and $L(r)$ is the image of the circle $|z|=r$, then the distance between $L(r)$ and E is at most $r-1$, with equality when E is a circle.

Suppose now that E is compact, and that $f(z)=b_0+b_1/z+\cdots$ is regular outside E and $|f(z)|\leqq 1$ there; and set $\alpha(E)=\max |b_1|$ subject to these hypotheses. Then $\alpha(E) \leqq$ cap E, with equality when E is connected. In the third paper the author proves that if C is the union of n continua $C_1, C_2, \cdots, C_n$ and S the Minkowski sum $S=C_1+C_2+\cdots+C_n$, then $\alpha(S)\geqq\alpha(C)$. He deduces that if E is a linear set, then $\alpha(E)=\frac{1}{4}L$, where L is the Lebesgue measure of E. The inequality $L/4 \leqq \alpha \leqq L/\pi$ was previously proved by Ahlfors and Beurling [Acta Math. **83** (1950), 101–129; MR **12**, 171]. Various other results are obtained, including one which shows that if $E_1, \cdots, E_n$ are continua which are distant a long way from each other then

$$\alpha(E_1 \cup E_2 \cup \cdots \cup E_n) \approx \alpha(E_1)+\alpha(E_2)+\cdots+\alpha(E_n),$$

whereas cap $(E_1 \cup E_2 \cup \cdots \cup E_n)$ becomes very large in this case. *W. K. Hayman* (London)

5741:

Jenkins, James A. On univalent functions with real coefficients. Ann. of Math. (2) **71** (1960), 1–15.

The author uses his general coefficient theorem [Trans. Amer. Math. Soc. **77** (1954), 262–280; MR **16**, 232] to study the class S_R of all normalized univalent functions in the unit circle with real coefficients. It is an immediate consequence of the $\frac{1}{4}$-theorem that the intersection of the image domains of all univalent normalized functions is the circle with radius $\frac{1}{4}$ around the origin. However, it is shown that the corresponding intersection for all functions of the class S_R is larger. This domain is doubly-symmetric and bounded by a curve with the equation in polar coordinates $r=m(\psi)$; $m(\psi)\geqq\frac{1}{4}$ and equality holds only for $\psi=0$, $\psi=\pi$. The function $m(\psi)$ can be calculated explicitly in terms of elliptic functions. Next, the set of all values $f(z_0)=Ae^{i\psi}$ is studied for fixed $|z_0|<1$ and all $f\in S_R$. Precise bounds $\alpha(\psi)\leqq A\leqq\beta(\psi)$ are established and the corresponding extremum functions are determined. This result includes various estimates for the class S_R given by Goluzin [Mat. Sb. (N.S.) **27** **(69)** (1950); 201–218; MR **12**, 490]. Finally, analogous but more complicated estimates for the derivatives of functions $f(z)\in S_R$ are given. *M. Schiffer* (Stanford, Calif.)

5742:

Hayman, W. K. On the coefficients of univalent functions. Proc. Cambridge Philos. Soc. **55** (1959), 373–374.

Let $f(z)=\sum a_nz^n$ and $g(z)=\sum b_nz^n$ be regular in $|z|<1$, and define their composition $f(z)\circ g(z)=\sum a_nb_n\cdot n^{-1}z^n$. It was conjectured that the univalency of f and g in $|z|<1$ implies the univalence of their composition in $|z|<1$. This conjecture was recently disproved by counterexamples: Epstein and Schoenberg [Bull. Amer. Math. Soc. **65** (1959), 273–275; MR **21** #7304]; Loewner and Netanyahu [ibid., 284–286; MR **21** #7305]. A new counterexample is now constructed as follows: Denote $f(z)=f_1(z)$ and define recursively $f_{p+1}(z)=f_p(z)\circ f(z)$. If $f(z)$ is univalent, the hypothesis would imply the univalence of all $f_p(z)$. Consider now $f(z)=(1-z)^{-b}-1$ with $b=1+e^{i\lambda}$. This function is univalent, but for $p\geqq 3$ and $(p-2)/p<\cos\lambda<1$ the function $f_p(z)$ is not. The latter fact is established by studying the logarithmic derivative of $f_p(z)$ near $z=1$ and showing that a well-known estimate for univalent functions is violated. This shows that the conjecture on composition functions is wrong. *M. Schiffer* (Stanford, Calif.)

5743:

Merkes, E. P.; Scott, W. T. Covering theorems for S-fractions. Math. Z. **73** (1960), 333–338.

Denote by K the set to which f belongs only in case f is a function from $|z|<1$ and there exists a complex number sequence $\{a_p\}_{p=1}^{\infty}$ such that $|a_p|\leqq\frac{1}{4}$ $(p=1, 2, \cdots)$ and

$$f(z) = \frac{z}{1}\,\frac{}{+}\,\frac{a_1z}{1}\,\frac{}{+}\,\frac{a_2z}{1}+\cdots.$$

Thale [Proc. Amer. Math. Soc. **7** (1956), 232–244; MR **17**, 1063] showed that each function f in K is univalent for $|z|<12\sqrt{2}-16$; and Perron [Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1956**, 233–236; MR **18**, 884] showed that there is no larger circular domain with center at the origin having this property. There is no domain properly including this circular domain in which all functions of K are univalent.

From the fact that, for any f in K, if $0<r\leqq\frac{1}{2}$ and $|z|\leqq 4r(1-r)$, then $|f(z)/z-1/[1-r^2]|\leqq r/[1-r^2]$, where the last inequality is sharp [Paydon and Wall, Duke Math. J. **9** (1942), 360–372; MR **3**, 297], the authors show that

the map of $|z|<12\sqrt{2}-16$ by $w=f(z)$ includes the domain $|w|<12-8\sqrt{2}$ and is included by the domain $|w|<4[\sqrt{2}-1]$ and these bounds are sharp. The result is improved for the functions of K such that the coefficient a_1 in the continued fraction is positive. It is shown that there exists a number B, with $0.938<B\leq 1$, such that the map of $|z|<1$ by any f in K univalently covers the interior of a circle of radius B. Finally, if u and v are omitted values of f for $|z|<4r(1-r)$ and $\operatorname{Arg} u-\operatorname{Arg} v=\pi$, then $|u-v|\geq 8r(1-r)/\sqrt{(1+r^2)}$.

H. S. Wall (Austin, Tex.)

5744:

Aleksandrov, I. A. Domains of definition of some functionals on the class of functions univalent and regular in a circle. Issledovaniya po sovremennym problemam teorii funkciĭ kompleksnogo peremennogo, pp. 39–45. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

Let S denote the class of functions $z=f(w)$ that are regular and univalent in $|w|<1$, with $f(0)=0$ and $f'(0)=1$. The author investigates the domain of variation of the functional

$$J=\frac{w^n(f'(w))^m}{f^n(w)|f(w)|^l}$$

for fixed w as f runs through the class S.

As a corollary he proves that if ω is a fixed point in $|w|<1$ and $f(w)\in S$ then f is convex in the circle with center at ω and radius $2-\sqrt{(3+|\omega|^2)}$, and this radius is sharp. With the same hypothesis, a noneuclidean circle with center ω and noneuclidean radius $\pi/2$ is mapped onto a domain that is starlike with respect to $f(\omega)$. Other results of this type are obtained.

A. W. Goodman (Lexington, Ky.)

5745:

Ulina, G. V. On the domains of values of certain systems of functionals in the classes of univalent functions. Vestnik Leningrad. Univ. **15** (1960), no. 1, 34–54. (Russian. English summary)

Let S be the class of regular univalent functions f in the unit disc, $f(z)=z+c_2z^2+\cdots$, and let $S(c_2)$ be the subclass of functions with given c_2. Jenkins [Ann. of Math. (2) **59** (1954), 490–504; MR **15**, 786] has determined the domain of values $(c_2, f(r))$, $r=|z|$, for $S(c_2)$ with $c_2\geq 0$. Using Goluzin's variational method, the author obtains the domain of values in the case when all coefficients of f are real. By the same method the domain of values $(\ln f(z)/z, \ln f'(z))$, which is independent of $\arg z$, is determined for the class S, extending some results of N. A. Lebedev [same Vestnik **10** (1955), no. 8, 29–41; no. 11, 3–21; MR **17**, 248, 599]. Another result concerns the domain of values for $(|f''(0)/f'(0)|, |f'(0)/F'(\infty)|)$ in the class of functions M mapping two non-overlapping domains onto the interior, resp. exterior, of the unit circle [cf. Lebedev, Dokl. Akad. Nauk SSSR **115** (1957), 1070–1073; MR **19**, 951].

V. Linis (Ottawa, Ont.)

5746:

Charzynski, Z.; Schiffer, M. A new proof of the Bieberbach conjecture for the fourth coefficient. Arch. Rational Mech. Anal. **5**, 187–193 (1960).

Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be regular and univalent in $|z|<1$. Then $|a_4|\leq 4$. This famous theorem was first proved by Garabedian and Schiffer [J. Rational Mech. Anal. **4** (1955), 427–465; MR **17**, 24]. The authors now give a much shorter and simpler proof of this theorem. The proof is based on two inequalities. (I) If $f(z)$ is as given above then

$$\Re\left\{a_4-2a_2a_3+\frac{13}{12}a_2^3+2l\left(a_3-\frac{3}{4}a_2^2\right)+l^2a_2\right\}\leq\frac{2}{3}+2|l|^2.$$

(II) With the same hypothesis $3|a_3-\frac{3}{4}a_2^2|\leq 4-|a_2|^2$.

The first inequality is implied, but not expressly stated in work of Grunsky [Math. Z. **45** (1939), 29–61], but the authors now give a new proof using variational methods. The second inequality is derived from the area theorem.

A. W. Goodman (Lexington, Ky.)

5747:

Komatu, Yûsaku. On a correspondence between classes of analytic functions with positive real part in annuli. Kōdai Math. Sem. Rep. **11** (1959), 147–157.

Let R denote the class of functions $\Phi(z)$ which are regular and have positive real part in $|z|<1$, normalized by the condition $\Phi(0)=1$. Let Rq denote the class of functions $\Phi(z;q)$ which are regular and have positive real part in the annulus $q<|z|<1$, normalized by the conditions $\operatorname{Re}\{\Phi(z;q)\}=1$, $|z|=q$ and $(2\pi)^{-1}\int_{-\pi}^{\pi}\Phi(qe^{i\phi};q)=1$.

The author proves several theorems establishing a correspondence in terms of series expansions, Herglotz type integrals and functional equations between the classes R and Rq. One of the more brief theorems is as follows: Let $\Phi(z)=1+\sum_{n=1}^{\infty}C_nz^n$ belong to R. Then, the function

$$\Phi(z;q)=1+\sum_{n=1}^{\infty}\frac{1}{1-q^{2n}}(C_nz^n-q^{2n}\bar{C}_nz^{-n})$$

belongs to Rq. Conversely, let $\Phi(z;q)=1+\sum_{n=-\infty}^{\infty}C_n(q)z^n$ belong to Rq. Then,

$$\Phi(z)=1+\sum_{n=1}^{\infty}(1-q^{2n})C_n(q)z^n$$

belongs to R.

W. C. Royster (Lexington, Ky.)

5748:

Eweida, M. T. On basic sets of polynomials. Proc. Math. Phys. Soc. Egypt. No. 22 (1958), 83–88 (1959). (Arabic summary)

Let the simple set $\{p_n(z)=\sum_0^n p_{nr}z^r\}$ $(n=0, 1, \cdots)$ of monic polynomials satisfy the conditions $|p_{nr}|\leq\binom{n}{r}$; then the corresponding basic series represent all entire functions of increase less than order one and type $\log_e 2$ $(=.6934)$ in every bounded region of the complex plane. Moreover, an example shows that the best possible type E_1 for basic series lies in the range $(.6934, .6944)$.

I. M. Sheffer (University Park, Pa.)

SPECIAL FUNCTIONS

See also 5587, 5613, 5696.

5749:

Mikusiński, J. The trigonometry of the differential equation $x''+x=0$. Wiadom. Mat. (2) **2**, 207–227 (1959). (Polish)

$W(t)=\frac{1}{3}e^{-t}-\frac{1}{3}e^{t/2}\cos\frac{1}{2}3^{1/2}t+3^{-1/2}e^{t/2}\sin\frac{1}{2}3^{1/2}t$ is a solution of

$$(1)\quad x'''+x=0;\quad x(0)=0,\quad x'(0)=0,\quad x''(0)=1.$$

$W'(t)$ satisfies (1) with initial conditions (0, 1, 0), and $W''(t)$ satisfies (1) with initial conditions (1, 0, 0). The "trigonometric" behavior of $W(t)$ refers to its satisfying various identities similar to the trigonometric ones. In particular, $W'''(t)=-W(t)$; $[W''(t)]^3-[W'(t)]^3+[W(t)]^3+3W''(t)W'(t)W(t)=1$; $W(t+u)=W(t)W''(u)+W'(t)W'(u)+W''(t)W(u)$; $W(-t)=[W'(t)]^2-W''(t)W(t)$. $W''(2^nt)$ and $W''(-2^nt)$ are given as polynomials in $W''(t)$ and $W''(-t)$. $W(t)$ is shown to be analytic in a neighborhood of 0 with $W(t)=\sum_{n=0}^{\infty}(-1)^{n+1}t^{3n-1}/(3n-1)!$. Tables of values are given for W, W', and W''. It is shown that W'' is a unique solution of the functional equation $f(2t)+2f(-t)=3[f(t)]^2$, $f''(0)=0$, $f'''(0)=-1$. An alternate proof of this fact is given by using binomial coefficients. The zeros of W and its derivatives are given by $w_n^{(m)}=\frac{1}{3}2\pi(n+\frac{1}{6}-\frac{1}{3}m)+b_n^{(m)}$; $m=0, 1, 2, \cdots$; $n=1, 2, \cdots$, and an estimate for the error $b_n^{(m)}$ is derived. A recurrence relation is used to get estimates of these for $n=1, 2, 3, 4$ and $m=0, 1, 2$. $w_1\approx 4.233219531$. It is shown that the distance between successive zeros of any solution of (1) is $\leq w_1$, but may be arbitrarily small. Variations induced by altering the initial conditions are considered.

S. Hoffman (Hartford, Conn.)

5750:

Mikusiński, J. Trigonometry of the differential equation $x^{(4)}+x=0$. Wiadom. Mat. (2) 4, 73–84 (1960). (Polish)

This is an extension of the author's paper reviewed above. $C(t)=\cosh 2^{-1/2}t\cos 2^{-1/2}t$ satisfies (1) $x^{(4)}+x=0$, as do $C'(t)$, $C''(t)$, and $C'''(t)$. If we let $S(t)=-C''(t)$, then the following "trigonometric" identities hold: $C^2-2C'S'-S^2=1$; $C(t+u)=C(t)C(u)+S'(t)C'(u)-S(t)S(u)+C'(t)S'(u)$; $C(4t)-2C(2t)[C(2t)+8C^2(t)]+32C^4(t)-16C^2(t)+1=0$ (and numerous others). $C(t)$ is the only function analytic in a neighborhood of 0 which satisfies the functional equation given by the last identity with $C(0)=1$, $C''(0)=0$, $C^{(4)}(0)=1$. Successive zeros of any solution of (1) differ by an amount $\leq 2(2\pi)^{1/2}$, but may be arbitrarily close to one another. Altering boundary conditions gives similar results.

S. Hoffman (Hartford, Conn.)

5751:

Angheluţă, Th. Sur une classe d'intégrales. Inst. Politehn. Cluj. Lucrări Şti. **1959**, 21–28. (Romanian. Russian and French summaries)

The author calculates the integrals

$$I_m=\int_0^1\frac{(\log x)^m}{1-x}\,dx,\quad K_m=\int_0^1\frac{(\log x)^m}{1+x}\,dx,$$

where m is a natural number. The results are obtained via applications of the theory of Bernoulli numbers and the calculus of residues.

E. R. Lorch (New York)

5752:

Darwin, Charles. Note on the algebra of elliptic functions. Quart. J. Mech. Appl. Math. **13** (1960), 129–137.

From the theta functions $\vartheta_r(x,q)$, $r=1, 2, 3, 4$, phi functions $\varphi_r(x,q)=\vartheta_r(x,q^2)$ are derived. The principle of the author's method for deriving all the well-known formulas in the theory of theta functions and the Jacobian elliptic functions is as follows. Take products of pairs of theta functions, and using two identities for changing the indices in a double summation, transform them into products of phi functions. For example $\vartheta_1'\vartheta_2=2\varphi_1'\varphi_4$, $\vartheta_3\vartheta_4=\varphi_4^2$, $\vartheta_2^2=2\varphi_2\varphi_3$ and therefore

$$f(q)=\frac{\vartheta_1'}{(\vartheta_2\vartheta_3\vartheta_4)}=\frac{\varphi_1'\varphi_4}{(\varphi_2\varphi_3\varphi_4)}=f(q^2)=f(q^{2^3})$$

$$=\lim_{q\to 0}f(q)=\lim_{q\to 0}\frac{2q^{1/4}}{(2q^{1/4}\cdot 1\cdot 1)}=1,$$

proving that $\vartheta_1'=\vartheta_2\vartheta_3\vartheta_4$. Apart from simplicity, the procedure allows each formula to be derived almost from first principles.

L. M. Milne-Thomson (Madison, Wis.)

5753:

Wahab, J. H. New cases of irreducibility for Legendre polynomials. II. Duke Math. J. **27** (1960), 481–482.

Some new cases of irreducibility of Legendre polynomials $P_n(x)$ are proven, for n of the form $n=\sum_{j=1}^{s}(p-1)p^{k_j}$ (with integral $k_1>k_2>\cdots>k_s\geq 0$), $p\leq 13$. As in the author's previous paper in same J. **19** (1952), 165–176 [MR **13**, 648], the proof uses Dumas' theorem. Assuming that $P_n(x)$ factors non-trivially over the rationals, $P_n(x)=p_m(x)p_r(x)$, $1\leq m\leq r<n$, Dumas' theorem (on the basis of Newton's polygon) assigns a lower bound b for m; in the cases here considered, b turns out to be higher than the known upper bound $n/17-1$ (see the author's quoted paper). The contradiction proves the irreducibility of $P_n(x)$. The paper also contains a counter-example to a lemma of Ille [Dissertation, Berlin, 1924].

E. Grosswald (Philadelphia, Pa.)

5754a:

Meulenbeld, B. Generalized Legendre's associated functions for real values of the argument numerically less than unity. Nederl. Akad. Wetensch. Proc. Ser. A. **61** = Indag. Math. **20** (1958), 557–563.

5754b:

Kuipers, L. Relations between contiguous generalized Legendre associated functions (recurrence formulas). Math. Scand. **6** (1958), 200–206.

5754c:

Kuipers, L.; Meulenbeld, B. Quadratic expansions of generalized Legendre's associated functions. J. London Math. Soc. **35** (1960), 221–224.

5754d:

Meulenbeld, Barend. Wronskians of linearly independent solutions of the generalized Legendre's equation. Recurrence formulas. Math. Nachr. **21** (1960), 193–200.

5754e:

Kuipers, L.; Robin, Louis. Résumé de quelques propriétés des fonctions de Legendre généralisées. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 502–507.

5754f:

Meulenbeld, B. On a generating function for the generalized Legendre's associated functions of the first kind. Nieuw Arch. Wisk (3) **7** (1959), 102–108.

5754g:

Meulenbeld, B. On derivatives of the generalized Legendre functions. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 200–206.

5754h:

Meulenbeld, B. New recurrence formulas for the $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$. Monatsh. Math. **64** (1960), 355–360.

The authors obtain further results on the generalized Legendre's associated functions $P^{m,n}(z)$ and $Q^{m,n}(z)$. These functions were defined in all points of the z-plane, except the part $(1, -\infty)$ of the real axis. [See Meulenbeld, Arch. Rational Mech. Anal. **3** (1959), 460–471; MR **21** #5032; and papers reviewed MR **21** #2073a–d.]

In a, functions $P_k^{m,n}(x)$ and $Q_k^{m,n}(x)$ are defined on the segment $-1<x<1$ of the real axis in a way analogous to the one usual for Legendre functions. Several relations for the above functions are derived.

In b, a set of three-term recurrence relations is derived for the contiguous functions $P_k^{m,n}$, $P_{k+1}^{m,n}$, $P_{k-1}^{m,n}$, $P_k^{m+1,n+1}$, $P_k^{m-1,n-1}$ for which $m-n$ is invariant. Of the 10 possible relations, 8 are given. {The reviewer checked (6), $\cdots$, (9) and found that the r.h. member of (9) should be preceded by a minus sign.} In (26) the author states a curious result, containing a hypergeometric function in the r.h. member, which, on inspection, turns out to be proportional to $P_k^{m,n+2}$. In effect (26) is a four-term recurrence relation, which by means of (8) is easily reduced to a three-term one. This latter formula is an example of a relation between functions for which $m-n$ is not invariant. {All given relations can be derived from a set of much simpler relations between the functions $P_k^{m,n}$, $P_{k\pm\frac{1}{2}}^{m\pm1,n}$ and $P_{k\pm\frac{1}{2}}^{m,n\pm1}$, as has been shown by the reviewer in his Bandung thesis.}

In c, the functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ are expanded in series of MacRobert's E-functions with $-z^2$ for argument.

In d, Wronskians are derived for the two sets $P_k^{m,n}(z)$, $Q_k^{m,n}(z)$ and $P_k^{m,n}(z)$, $P_k^{n,m}(-z)$. In the second part of this paper the derivative of $P_k^{m,n}$ is expressed in terms of $P_k^{m,n}$ and $P_{k-1}^{m,n}$, and similarly for $Q_k^{m,n}$. A few relations already obtained in c are rederived.

In e, the connection between Jacobi polynomials and generalized associated Legendre functions is used to translate results in terms of the former functions into the latter, notably: the normalization integral, a recurrence relation, some limits for $x\to\pm1$, including a generalization of a formula of Watson, a generalization of Rodrigues' formula, the differential equation, a recurrence relation involving a derivative, expression in terms of the hypergeometric function, a generating function and an integral representation.

In f, a generalization of a well-known generating function for Legendre polynomials and Gegenbauer functions is derived.

In g, formulae are found for the derivatives of $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ with respect to z, k, m and n. [This paper is also reviewed in MR **22** #1700.]

In h, it is shown that $P_k^{m,n}(z)$ is proportional to $Q_{-(1+m)/2}^{-2k-1,n}((z+3)/(z-1))$. The paper furthermore contains a rather haphazard collection of relations between the functions $R_{k\pm\frac{1}{2}}^{m,n\pm1}$ and $R_k^{m\pm2,n}$, R denoting P or Q.

D. J. Hofsommer (Amsterdam)

5755:

MacRobert, T. M. The multiplication formula for the gamma function and E-function series. Math. Ann. **139**, 133–139 (1959).

The multiplication formula for the gamma function, Barnes' integral for E-functions, and several theorems yielding the values of ${}_pF_q$'s of unit argument are used to sum various series of E-functions.

N. D. Kazarinoff (Moscow)

5756:

MacRobert, Thomas M. Expression for an E-function as a finite series of E-functions. Math. Ann. **140** (1960), 414–416.

If m is a positive integer,

$$E(p;\alpha_r:q;\rho_s:z) =$$

$$\sum_{t=0}^{m-1}(-1)^t(n+2t)z^{-t}E\begin{pmatrix} n+t,\ \alpha_1+t,\ \cdots,\ \alpha_p+t:z \\ n+1+2t,\ \rho_1+t,\ \cdots,\ \rho_q+t\end{pmatrix}$$

$$+(-z)^{-m}E\begin{pmatrix} n+m,\ \alpha_1+m,\ \cdots,\ \alpha_p+m:z \\ n+2m,\ \rho_1+m,\ \cdots,\ \rho_q+m\end{pmatrix}.$$

A special case of this result, for example, leads, when $m\to\infty$, to the formula

$$\tfrac{1}{2}zJ_n(z) = \sum_{t=0}^{\infty}(-1)^t(n+1+2t)J_{n+1+2t}(z).$$

N. D. Kazarinoff (Moscow)

5757:

Bacon, Ralph Hoyt. A note concerning orthogonal polynomials. SIAM Rev. **2** (1960), 269–276.

Polynomials orthogonal with weight one on any finite interval (a, b) are in fact Legendre polynomials transformed for this interval. So are the five cases considered by the author. The first set is for instance Legendre polynomials $L_n(t)$ multiplied by the numerical factor $2^{-n}T^n\cdot n!/(2n-1)!!$, where the variable t was replaced by $x=T(1+t)/2$.

The idea itself of transforming the classical known polynomials to the interval $(0, T)$ is useless: it is much simpler to transform the interval, introducing another variable, so that the new interval is $(-1, 1)$.

E. Kogbetliantz (New York)

5758:

Koschmieder, Lothar. Turánsche und Forsythesche Funktionenfolgen. Monatsh. Math. **64** (1960), 263–271.

The sequence of functions $\{\varphi_n(x)\}$ is a Turán sequence in some interval I if $\varphi_{n+1}\varphi_{n-1}-\varphi_n^2<0$ in I and a Forsythe sequence in some interval J if $\varphi_{n+2}\varphi_{n-1}-\varphi_{n+1}\varphi_n<0$ in J. The following theorem is proved: If $\{\varphi_n(x)\}$ is a Turán sequence in I and a Forsythe sequence in J, then $\{\varphi_n(x)+\varphi_{n+1}(x)\}$ is a Turán sequence in $I\cap J$. It is known that the Legendre polynomials $P_n(x)$ and the Tschebyscheff polynomials of the first and second kind $T_n(x)$ and $U_n(x)$ are Turán sequences in $-1<x<1$ and Forsythe sequences in $0<x<1$ (these are special cases of ultraspherical polynomials which also have these properties; cf. reviewer's paper in Duke Math. J. **26** (1959), 349–359 [MR **21** #3594]), and therefore $\{\varphi_n(x)+\varphi_{n+1}(x)\}$, with $\varphi_n(x)$ equal to $P_n(x)$, $T_n(x)$, and $U_n(x)$ in turn, are Turán sequences in $0<x<1$. The sequences $\{\tanh nx\}$, $\{\operatorname{sech} nx\}$, and the elliptic functions $\{\operatorname{sn} nx\}$, $\{\operatorname{cn} nx\}$ are similarly treated.

A. E. Danese (Schenectady, N.Y.)

5759:

Vilenkin, N. Ya. Deduction of certain properties of Jacobi polynomials from the theory of group representation. Moskov. Gos. Ped. Inst. Uč. Zap. **108** (1957), 59–71. (Russian)

In this paper the theory of matrix representations of the group of rotations of 3-dimensional space is applied to obtain a number of properties of the Jacobi polynomials. Most relations obtained in this way are well-known but some might be new. This paper complements the extremely readable paper of Gel'fand and Šapiro in Uspehi Mat. Nauk **7** (1952), no. 1 (47), 3–117; Amer. Math. Soc. Transl. (2) **2** (1956), 207–316 [MR **13**, 911; **17**, 875].

H. A. Lauwerier (Amsterdam)

ORDINARY DIFFERENTIAL EQUATIONS

See also 5825, 5986, B6640.

5760:

Collatz, Lothar. ★Differentialgleichungen für Ingenieure: Eine Einführung. 2., neubearbeitete und erweiterte Aufl. Leitfäden der angewandten Mathematik und Mechanik, Bd. 1. B. G. Teubner Verlagsgesellschaft, Stuttgart, 1960. 197 pp. DM 21.60.

This small book gives a wealth of information on differential equations. It contains, indeed, more than can be dealt with effectively in a general course on differential equations in engineering curricula. The concise formulation is clear and precise, and it will appeal to engineers on account of its many graphic examples. The chapter titles are: Ordinary differential equations of the first order; Ordinary differential equations of higher order; Boundary value and eigenvalue problems; Special differential equations; Miscellaneous additional problems. Nonlinear problems receive more attention than is customary in books of this type. Reviewer recommends the book to all engineers who encounter differential equations.

W. T. Koiter (Delft)

5761:

Agnew, Ralph Palmer. ★Differential equations. 2nd ed. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1960. ix+485 pp. $7.50.

This second edition of a well-known text is much more than a minor revision. Intended as both a text and a reference book, it gives valuable preliminary glimpses of matters encountered in advanced mathematics and science. Problems are discussed at greater length and more accurately than is usual, with a resultant difference in emphasis. A considerable part of a student's time is to be spent reading the text and answering the questions that so arise. The rules of the game are clearly stated: "We work some of the problems and read all of them." More attention than usual is given to the derivation of differential equations. This edition like the first has interesting, or humorous, side remarks. Particularly valuable are those which explain mathematical idiom or which show the relation of mathematics to physics. The problems are excellently chosen to bring out the power of mathematical analysis. Particularly for the beginning applied mathematician, analyst, or scientist this book can provide a deeper understanding than most.

There are sixteen chapters, of which the first seven are suggested for a one semester course. There is some mention of partial differential equations, but no systematic study. The following selective comments mention some of the significant features or departures from the first edition. The introductory chapter 1 contains an elegant derivation of the differential equation of a mass-spring system. Chapter 2 is devoted to applications of the fundamental theorem of calculus to differential equations. A feature is the full treatment of keplerian motion. Chapter 3 is on linear first order. Chapter 4 discusses families of curves and general and singular solutions. It is meant to be read but not dwelt upon. Chapter 5 is on first order equations. Chapter 6, 98 pp., on linear equations, contains a wealth of material in the problems. Chapter 7, on series, has some new material, for example, a long problem on the gamma function. Chapter 8 is a new and elegant treatment of numerical methods. One would hope that part of it could be included in a first course. Chapter 9 on the Laplace transform is new.

This is a well-written book which merits study by any serious student. Hopefully, it will have an influence on the authors of other books. *M. E. Shanks* (Chapel Hill, N.C.)

5762:

Murphy, George M. ★Ordinary differential equations and their solutions. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960. ix+451 pp. $8.50.

The person who must find explicit solutions to special differential equations will find this work a useful companion to the well-known book by E. Kamke. It is devoted entirely to methods for finding solutions, and consists of two parts.

Part 1 (222 pages) describes the standard devices for finding solutions, usually general solutions. The exposition is classical in terminology. For example, general, particular, and singular solutions are described in the customary vague way. Nevertheless, to one interested in special cases this will suffice. The reader who is not a mathematician may find the book easier to read than that of Kamke. The headings and subheadings, e.g., first order, first order and higher degree, etc. are similar to those in Kamke, though the latter has more material. For example, Kamke discusses Green's functions, boundary value problems, and the Runge-Kutta method, whereas the author does not. Yet for most users the coverage is adequate.

Part II (208 pages) gives solutions to 2315 equations (some 1500 are given by Kamke). Naturally, the author faced a serious problem of selection and, though many more could have been included, his choice includes enough special types involving "tricks" to suggest other devices to the reader. The search by the user for a particular type is made easier by a special index which lists, under the various section headings of part I, the numbers of all the solutions coming under each heading. Undoubtedly this book will be widely used by chemists, physicists, and engineers. *M. E. Shanks* (Chapel Hill, N.C.)

5763:

Bandić, I. Über die Integration der linearen Differentialgleichungen in geschlossener Form. Bull. Soc. Math. Phys. Macédoine **9** (1958), 21–30. (Serbo-Croatian. German summary)

The author considers the problem of the reduction of the linear differential equation $\sum_{i=1}^{n} a_i(x)y^{(n-i)}=0$ to lower order, and the possibility of solving it in finite form. Using the relative derivatives $\Delta_n(u)$ $(\Delta_n(u)=u^{(n)}/u)$, he gives the conditions on the coefficients of the equation in order for it to be integrable. The paper contains a detailed investigation for the cases $n=3$ and $n=4$.

B. S. Popov (Skopje)

5764:

Nevanlinna, Rolf. Cauchy's polygon method. Arkhimedes **1958**, no. 2, 1–11. (Finnish)

The author discusses the use of Cauchy's polygon method in the existence and uniqueness proofs of the solution of the differential equation $dy/dx=f(x, y)$, where $f(x, y)$ is continuous in $|x|\leqq\rho_x$, $|y|\leqq\rho_y$. It is assumed, moreover, that

$$(1)\qquad \int_0^{\rho_y}\frac{d\rho}{\psi(\rho)}=\infty$$

with $\psi(\rho)=\sup|f(x, y_1)-f(x, y_2)|$ for $|x|\leqq\rho_x$, $|y_i|\leqq\rho_y$ $(i=1, 2)$, $|y_1-y_2|\leqq\rho$. Then it is shown that Cauchy's polygon $y=y_D(x)$ which corresponds to the subdivision D of $|x|\leqq\rho_x$ approaches a limiting function $y=y(x)$ with $y(0)=0$ which satisfies $y'=f(x, y)$ for $|x|<\rho_x$. The uniqueness of the solution was shown by Osgood in 1899.

The generality of the theorem is in that (1) is trivially fulfilled if f satisfies the Lipschitz condition usually used in the theory of differential equations. Condition (1) is, more generally, satisfied by the function

$$\psi(\rho)=C\rho\log\frac{1}{\rho}\log\log\frac{1}{\rho}\cdots\log_n\frac{1}{\rho},$$

but replacing the last factor by $(\log_n 1/\rho)^{1+\epsilon}$ already suffices to make integral (1) convergent.

The author's approach leads to existence and uniqueness from a single starting point and gives an explicit estimate showing the rate of convergence of Cauchy's method. The argument can be generalized to finite and infinite systems of differential equations.

L. Sario (Los Angeles, Calif.)

5765:

Mrówka, S. The fixed-point theorem and its application to the theory of differential equations. Wiadom. Mat. (2) **2**, 292–297 (1959). (Polish)

The existence of solutions to $y'=\varphi(t, y)$, $y(0)=0$, where φ is continuous on $[0, 1]\times[-C, C]$, is proved by using the fact that the set of all functions satisfying a Lipschitz condition (with constant C) on $[0, 1]$, and vanishing at 0, is homeomorphic to a closed convex subset of the Hilbert cube.

S. Hoffman (Hartford, Conn.)

5766:

Corduneanu, C. Sur un théorème de Perron. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) **5** (1959), 33–36. (Russian and Romanian summaries)

Let $M[\tilde{M}]$ denote the space of mappings u of $J=[0, \infty)$ into R^n which are integrable on every compact subinterval of J and such that $\int_t^{t+1}\|u\|dt$ is bounded for every $t\in J$. Let $x'=A(t)x+f(t, x)$ where $x\in R^n$, and suppose (i) $A\in\tilde{M}$; (ii) f is measurable in t, continuous in x, for $t\in J$ and $\|x\|\leqq a$; (iii) $\|f(t, x)\|\leqq\varphi(t)$ for $t\in J$ and $\|x\|\leqq a$, where $\varphi\in M$ and $\|\varphi\|_M$ is sufficiently small; and (iv) the equation $x'=A(t)x+u(t)$ has at least one bounded solution in J for every $u\in M$. Under these assumptions the author proves, by use of the Schauder-Tychonoff fixed point theorem, that there exists at least one solution in J with $\|x(t)\|\leqq a$. In the more precise results of Massera and Schäffer [Ann. of Math. (2) **67** (1958), 517–573; MR **20** #3466], f is required to satisfy a Lipschitz condition in place of (iii).

H. A. Antosiewicz (Los Angeles, Calif.)

5767:

Sarantopoulos, Spyridon. Sur l'existence des intégrales holomorphes des équations différentielles du premier ordre dans le cas singulier. Bull. Soc. Math. Grèce **28** (1954), 128–166.

L'A. considera l'equazione differenziale (1) $x^2dy/dx=\alpha(x)y+x\varphi(x)+xy\sigma(x)$, dove $\varphi(x)=\sum_{\nu=0}^{\infty}\beta_\nu x^\nu$, $\sigma(x)=\sum_{\nu=0}^{\infty}\delta_{\nu+1}x^\nu$ sono olomorfe nell'intorno dell'origine, e supposto che (2) $y=\sum_{\nu=0}^{\infty}\gamma_{\nu+1}x^{\nu+1}$ sia una soluzione formale della (1), nel caso che δ_1 non sia intero positivo o nullo, esprime i coefficienti $\gamma_{\nu+1}$ con dei determinanti di cui studia in questa prima parte alcune proprietà che gli occorreranno per dare le condizioni perchè la (2) rappresenti una funzione olomorfa.

G. Sansone (Zbl **56**, 309)

5768:

Sarantopoulos, Spyridon. Sur l'existence des intégrales holomorphes des équations différentielles du premier ordre dans le cas singulier. II. Bull. Soc. Math. Grèce **29**, 1–24 (1954).

L'A. completa una precedente ricerca [si veda la precedente recensione] e dimostra successivamente che data l'equazione differenziale

$$x^{\mu+1}dy/dx=xy+x\varphi(x)+\delta x^\mu y,$$

con μ intero, $\mu\geqq 1$, $\varphi(x)=\beta_0+\beta_1x+\beta_2x^2+\cdots$, olomorfa in un intorno di $x=0$, e supposto che δ non sia intero positivo, allora condizione necessaria e sufficiente perchè questa possieda un integrale $y(x)$ olomorfo in un intorno di $x=0$, con $y(0)=0$, è che risulti $\Phi_\nu(x)=0$ $(\nu=0, 1, \cdots, \mu-1)$ essendo la trascendente intera $\Phi_\nu(x)$ definita della relazione

$$\Phi_\nu(x)=\sum_{\lambda=0}^{\infty}\beta_{\mu\lambda+\nu}x^\lambda/(\nu+1-\delta)(\nu+1+\mu-\delta)\cdots(\nu+1+\mu(\lambda-1)-\delta).$$

Se poi δ è un numero intero positivo, e $\delta=h+1+\mu(l-1)$, alla condizione $\Phi_h(x)=0$ deve invece sostituirsi l'altra $\varphi_h(\alpha)=0$, dove

$$\varphi_h(x)=\sum_{k=0}^{\infty}\beta_{\mu(l+k)+h}x^k/k!\mu^k.$$

Tanto nell'uno che nell'altro caso il raggio di convergenza di $y(x)$ non è inferiore al raggio di convergenza di $\varphi(x)$.

G. Sansone (Zbl **58**, 311)

5769:

Tereščenko, N. I. Finite form of solutions of a system of linear differential equations with polynomial coefficients. Ukrain. Mat. Ž. **11** (1959), 93–104. (Russian. English summary)

Perron [Acta Math. **34** (1911), 139–163] first treated the problem mentioned in the title of the present paper. Later Latyševa [same Ž. **1** (1949), no. 3, 81–100; MR **13**, 944] and the author [ibid. **10** (1958), no. 2, 220–223; MR **20** #1812] made various extensions. In the present

paper the author finds necessary and sufficient conditions that the system (1) $t\sum_{i=0}^{k} b_i t^i \dot{y} = \sum_{j=0}^{l} A_j t^j y$ (where y is an n-vector, the b_i's are constant diagonal and the A_j's constant square matrices) have a solution of the form $y = t^r(c_0 + c_1 t + \cdots + c_q t^q)$. The conditions, too complicated to give here, are obtained by replacing y in (1) by $t^r c$, c a constant vector, ordering the terms of (1) according to powers of t, and then analyzing the coefficients of these powers. The author also gives an algorithm for finding finite solutions of (1). *C. S. Coleman* (Claremont, Calif.)

5770:

Hladík, František. Note on solutions of an equation $y'' + fy' + \varphi y = 0$ one of whose particular solutions is the derivative of another. Časopis Pěst. Mat. **85** (1960), 202–203. (Czech)

In the equation in the title, let f, φ have continuous derivatives of second order in an open interval J; let $f'(x) \neq 0$, $x \in J$. It is shown that the given equation possesses a particular solution A, with $B = A'$ also a solution, if and only if f, φ satisfy in J the equation (1) $u' + fu + u^2 + \varphi = 0$, where $u = -\varphi'/f'$. Moreover, if (1) is satisfied $A = \exp(\int u\,dx)$. In case $y'' - 4xy' + (4x^2 - 2)y = 0$, (1) holds with $u(x) = 2x$, and $A = \exp(x^2)$, $B = 2x\exp(x^2)$. *J. A. Nohel* (Atlanta, Ga.)

5771:

Borůvka, M. O. [Borůvka, O.] Sur les transformations différentielles linéaires complètes du second ordre. Ann. Mat. Pura Appl. (4) **49** (1960), 229–251.

The results of a previous paper [same Ann. (4) **41** (1956), 325–342; MR **20** #1814] are only of a local character, i.e., the solutions of (b) exist (and hence the transformation (13) applies) only in intervals which are smaller than the intervals j, J of definition of (a), (A). A solution of (b) is called complete if it is defined on j and its values cover J; the corresponding transformation is also called complete. The present work is devoted to the investigation of the existence of complete solutions. The decisive condition is that (a), (A) have the same type m (supposed to be finite and ≥ 2), the type being the maximum number of zeros of the solutions in the interval of definition, and are both simultaneously special or non-special, an equation (a) being special if $\inf\{t \in j;\, t$ has a conjugate in j which is $< t\}$ is conjugate to $\sup\{t \in j;\, t$ has a conjugate in j which is $> t\}$. A detailed description of the results would be too lengthy to be reproduced here. *J. L. Massera* (Montevideo)

5772:

Rabinovič, Yu. L. Behavior at infinity of solutions of second-order linear differential equations. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 5, 53–63. (Russian)

One of the results is the following theorem. If the coefficients of (1) $y'' + 2p(x)y' + q(x)y = 0$ satisfy for $x \geq x_0$ the inequalities $|p(x) + (\rho - 1)/2x| \leq Px^{\rho-1}$, $|q(x)| \leq Qx^{2\rho-2}$, where P, Q and $\rho > 0$ are constants, then it holds for every solution $y(x)$ of (1)

$$A\exp\{-\lambda_0(x^\rho - x_0^\rho)/\rho\} < \alpha_0|y(x)| + |y'(x)|/x^{\rho-1} < A\exp\{\lambda_0(x^\rho - x_0^\rho)/\rho\},$$

where $\lambda_0 = \sqrt{(P^2 + Q)} + P$, $A = \alpha_0(|y(x_0)| + \lambda_0|y'(x_0)|/Qx_0^{\rho-1})$. Another result is represented by asymptotic formulas for the solutions of $y'' + q(x)y = 0$, where $q(x) \simeq a_0 x^\mu$ for $x \to \infty$. The author obtains four different formulas according to $\mu < -2$, $\mu = -2$ and $a_0 < \frac{1}{4}$, $\mu = -2$ and $a_0 > \frac{1}{4}$, $\mu > -2$. *M. Zlámal* (Brno)

5773:

Ráb, Miloš. Kriterien für die Oszillation der Lösungen der Differentialgleichung $[p(x)y']' + q(x)y = 0$. Časopis Pěst. Mat. **84** (1959), 335–370; erratum **85** (1960), 91. (Czech and Russian summaries)

Let $p'(x)$, $q(x)$ be continuous, and suppose $p(x) > 0$ on $J = [x_0, \infty)$. A solution $y(x) \not\equiv 0$ of (1) $(p(x)y')' + q(x)y = 0$ is said to be oscillatory if it has an infinity of zeros on J, the equation is said to be oscillatory if every solution is. In section 1, the method whereby (1) is transformed into a Riccati equation is used to prove the main theorem, namely that the equation (1) is oscillatory if and only if there is a function $g(x) \in C^1$, $g(x) > 0$ in J, for which

$$\lim_{x\to\infty} \int_{x_0}^{x} \frac{1}{p(x_1)} \exp\left\{2\int_{x_0}^{x}\left[\frac{1}{p(s)g^2(s)} \int_{x_0}^{s}(q(t)g^2(t) - p(t)g'^2(t))\,dt - a\right]ds\right\}dx_1 = \infty,$$

holds for every constant a. There are then obtained many known sufficient conditions in order that (1) be oscillatory. In section 2 the method of the introduction of polar coordinates [Prüfer, Math. Ann. **95** (1926), 499–518] and the use of approximate limits [Olech, Opial and Ważewski, Bull. Acad. Polon. Sci. Cl. III **5** (1957), 621–626; MR **19**, 650] are used to obtain sufficient conditions in order that (1) be oscillatory. *C. R. Putnam* (Lafayette, Ind.)

5774:

Chandrasekhar, S. Magnetohydrodynamics. Frontiers of numerical mathematics, pp. 99–106. University of Wisconsin Press, Madison, Wis., 1960.

Mathematical problems arising in three unrelated fluid-mechanical problems are briefly discussed here; only the first is in the area of magnetohydrodynamics. This is, specifically, the problem of gravitational equilibrium of an incompressible fluid mass with axisymmetric fluid motions and magnetic fields. A formal solution involves several arbitrary functions and is therefore of limited utility. A variational solution is suggested but not carried out here. The second problem arises in investigations of hydrodynamic stability and involves variational formulation of a non-self-adjoint characteristic-value problem in a high-order differential equation. The given problem has the same characteristic values as its adjoint, and there is an integral condition involving the solutions u_i of the original and ϕ_j of the adjoint for $i \neq j$. The third problem arises in studying stability of Couette flow; it is a characteristic-value problem in which the characteristic-value parameter n occurs nonlinearly. It is shown how conclusions can be drawn about the stability if the problem is reformulated as a linear characteristic-value problem for another parameter occurring in the problem, as if n were known. *W. R. Sears* (Ithaca, N.Y.)

5775:

Coppel, W. A. On a differential equation of boundary-layer theory. Philos. Trans. Roy. Soc. London. Ser. A **253** (1960), 101–136.

The author establishes the following result. The boundary-value problem

$$y''' + yy'' + \lambda(1-y'^2) = 0,$$
$$y = \alpha,\quad y' = \beta \text{ at } x = 0;\quad y' \to 1 \text{ as } x \to \infty$$

has a solution for any non-negative values of the constants α, β. The second derivative y'' is positive, zero, or negative throughout the interval $0 \leq x < \infty$ according as β is less than, equal to, or greater than 1. He shows that with this restriction on y'' the solution of the problem is unique. This result is a generalization of Weyl's result for $\alpha = \beta = 0$ [Ann. of Math. (2) **43** (1942), 381–407; MR **3**, 284]. A detailed analysis is presented for the properties of all solutions of the equation. The case $\lambda = 0$ is discussed separately. *B. S. Popov* (Skopje)

5776:

Levitan, B. M.; Sargsyan, I. S. Some problems in the theory of the Sturm-Liouville equation. Uspehi Mat. Nauk **15** (1960), no. 1 (91), 3–98 (Russian); translated as Russian Math. Surveys **15**, 1–95.

This extensive study is devoted to the expansion of a given function in the eigenfunctions of $y'' + \{\lambda - q(x)\}y = 0$ valid in a finite or infinite interval. By using the method sketched briefly below, the authors are able to obtain a number of sharp results, thus improving upon well-known results. Let the spectrum be discrete with eigenvalues λ_n and eigenfunctions $\psi_n(x)$ $(n = 1, 2, 3, \cdots)$. Then they start with considering the Cauchy problem $\partial^2 u/\partial x^2 - q(x)u = \partial^2 u/\partial t^2$, $u = f(x)$ and $u_t = 0$ for $t = 0$. This problem may be solved in two distinct ways: (a) by the Fourier method; (b) by the method of Riemann. By equating the different forms in which the (unique) solution is obtained, an identity is obtained which provides a fruitful basis for further investigations. Among other things, some Tauberian theorems of the Fourier integrals are applied. One of the results obtained here is that a quadratically integrable function can be expanded in a uniformly convergent series with the usual Fourier representation of its coefficients. The authors also study the differentiability of such expansions. The same method is applied to the study of the asymptotic properties of the spectral function of the problem. *H. A. Lauwerier* (Amsterdam)

5777:

McLeod, J. B.; Titchmarsh, E. C. On the asymptotic distribution of eigenvalues. Quart. J. Math. Oxford Ser. (2) **10** (1959), 313–320.

This paper contains simpler proofs of results proved by Titchmarsh in two earlier papers, viz., same J. (2) **5** (1954), 228–240; Proc. Roy. Soc. London Ser. A **245** (1958), 147–155 [MR **16**, 824; **20** #2530]. *E. T. Copson* (St. Andrews)

5778:

Lykova, O. B. On the stability of solutions of systems of nonlinear differential equations. Ukrain. Mat. Ž. **11** (1959), 251–255. (Russian. English summary)

Author's summary: "A theorem is proved about the stability of the trivial solution $x = 0$ of $\dot{x} = X(x)$ acted on by a slight perturbation characterized by functions $\varepsilon X^*(t, x, \varepsilon)$ in the critical case of a pair of imaginary roots of the Jacobian matrix $X_x(0)$ (the other roots having negative real parts)." *J. L. Massera* (Montevideo)

5779:

Opial, Z. Sur la stabilité asymptotique des solutions d'un système d'équations différentielles. Ann. Polon. Math. **7** (1960), 259–267.

Let $x' = f(x, t)$ be a 2-dimensional system with f of class C^1 on E^3, and suppose that there exists a compact simply connected set $K \subset E^2$ such that, for every $x_0 \in K$ and every t_0, $x(t, x_0, t_0) \in K$ for $t \geq t_0$. The author shows that if the variation equations relative to every such solution are asymptotically stable then, for every $x_1, x_2 \in K$ and every t_0, $x(t, x_1, t_0) \to x(t, x_2, t_0)$ as $t \to \infty$. This modifies a general stability criterion of Seifert [Ann. of Math. (2) **67** (1958), 83–89; MR **19**, 960] and yields a simpler proof, under weaker hypotheses, for a theorem of the latter concerning the stability of a periodic solution of the equation $x'' + f(x)x' + g(x) = p(t)$ with p periodic. The author also gives some other applications, using results of Reuter [Proc. Cambridge Philos. Soc. **47** (1951), 49–54; MR **12**, 827] and of Cartwright and Littlewood [Ann. of Math. (2) **48** (1947), 472–494; MR **9**, 35] on the existence of sets such as K. *H. A. Antosiewicz* (Los Angeles, Calif.)

5780:

Opial, Z. Sur les solutions périodiques et presque-périodiques de l'équation différentielle $x'' + kf(x)x' + g(x) = kp(t)$. Ann. Polon. Math. **7** (1960), 309–319.

Suppose that in the equation of the title f, g, p are continuous everywhere and such that: (i) $f(x) > 0$, $\operatorname{sgn} x \int_0^x f(u)du \to \infty$ with $|x|$; (ii) $xg(x) > 0$ for $x \neq 0$, $\int_0^x g(u)du \to \infty$ with $|x|$, and $|p(t)| \leq P$, $|\int_0^t p(s)ds| \leq P$. Under these conditions Reuter [Proc. Cambridge Philos. Soc. **47** (1951), 49–54; MR **12**, 827] proved that there exist positive constants x_0, y_0 independent of k such that, for every k and every solution, $|x(t)| \leq x_0$, $|x'(t)| \leq y_0$ for t sufficiently large. Applying a method he had used earlier [see the paper reviewed above and also Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 495–500; MR **22** #2762], the present author proves a comparison theorem which implies the following result. If g is of class C^2 on $|x| \leq x_0$, $g'(x) > 0$, and $k > \frac{1}{2}y_0 \max_{|x| \leq x_0} \{|g''(x)|/f(x)g'(x)\}$ then, for every two solutions, $x_1(t) \to x_2(t)$, $x_1'(t) \to x_2'(t)$ as $t \to \infty$. This improves on a theorem of Cartwright and Littlewood [Ann. of Math. (2) **48** (1947), 472–494; MR **9**, 35]. It also shows that if p is periodic [almost periodic] there exists a unique periodic [almost periodic] solution which is asymptotically stable. *H. A. Antosiewicz* (Los Angeles, Calif.)

5781:

Opial, Z. Sur l'allure asymptotique des solutions de certaines équations différentielles de la mécanique non linéaire. Ann. Polon. Math. **8** (1960), 105–124.

The author considers equations $x' = f(t, x) + g(t, x)$ with f, g continuous in $R \times R^n$, for which he assumes the existence of a non-negative valued function V of class C^1 in R^n such that $V \to \infty$ with $\|x\|$ and $f(t, x) \cdot \operatorname{grad} V(x) \leq 0$ in $R \times R^n$. If, further, $|g(t, x) \cdot \operatorname{grad} V(x)| \leq \rho(t, V(x))$ in

$R \times R^n$ for some continuous function ρ in $R \times R_+$, then boundedness of every solution of $u' = \rho(t, u)$ implies that every solution $x(t)$ is also bounded. From this the author derives, by specialization of ρ, g and f, extensions of various boundedness and stability theorems [see, e.g., H. A. Antosiewicz, J. London Math. Soc. **30** (1955), 64–67; MR **16**, 477; P. Santoro, Boll. Un. Mat. Ital. **11** (1956), 432; MR **18**, 308; G. Sestini, Riv. Mat. Univ. Parma **5** (1954), 227–232; MR **17**, 263; Z. Opial, same Ann. **8** (1960), 65–69; MR **22** #3850].

H. A. Antosiewicz (Los Angeles, Calif.)

5782:

Blinčevskiĭ, V. S. Existence of a periodic solution for an autonomous system of n differential equations. Mat. Sb. (N.S.) **50 (92)** (1960), 117–126. (Russian)

Suppose an autonomous system of n differential equations defines a vector field on a neighborhood of a torus T such that every vector (of small enough length) issuing from a point on the boundary of T either points into the interior of T or is tangent to T. The author proves that T is the union of non-empty closed connected invariant sets and the set of positive half-trajectories issuing from the boundary of T. From this he derives a sufficient condition for the existence of a closed trajectory.

H. A. Antosiewicz (Los Angeles, Calif.)

5783:

Čžan, Pan-gin′. On stability with arbitrary initial perturbations of the solutions of a system of two differential equations. Acta Math. Sinica **9** (1959), 442–445. (Chinese. Russian summary)

The trivial solution of the system

$$(1) \qquad \frac{dx_i}{dt} = X_i(x_1, x_2, \cdots, x_n),$$

where $X_i(0, 0, \cdots, 0) = 0$ $(i = 1, 2, \cdots, n)$, is called stable in the large if every other solution tends to 0 as $t \to \infty$. It is known that the trivial solution of (1) is stable in the large if there exists a positive definite function $V(x_1, x_2, \cdots, x_n)$ which tends to ∞ as $\sum_{i=1}^n x_i^2 \to \infty$, such that dV/dt is negative definite along solutions of (1). [See Barbašin and Krasovskiĭ, Dokl. Akad. Nauk SSSR **86** (1952), 453–456; MR **14**, 646.] Applying this result to the system

$$(2) \qquad \frac{dx}{dt} = f_1(x) + g_1(y), \quad \frac{dy}{dt} = f_2(x) + g_2(y),$$

where $f_i(0) = g_i(0) = 0$, the author gives several explicit criteria. The following one is typical: If $\int_0^t g_1(t)dt$ and $-\int_0^t f_2(t)dt$ are positive for $t \neq 0$ and tend to ∞ as $|t| \to \infty$, if $g_1(y)g_2(y) - f_1(x)f_2(x) < 0$ for $(x, y) \neq (0, 0)$, then the trivial solution of (2) is stable in the large.

Choy-tak Taam (Washington, D.C.)

5784:

Tumarkin, S. A. Asymptotic solution of a linear nonhomogeneous second order differential equation with a transition point and its application to the computations of toroidal shells and propeller blades. Prikl. Mat. Meh. **23** (1959), 1083–1094 (Russian); translated as J. Appl. Math. Mech. **23**, 1549–1565.

The form of equation considered is

$$\varepsilon \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \varepsilon r(x)]y = f(x);$$

ε is a small parameter, real or complex, and x ranges in a real interval (a, b). The functions p, q, r are real, $p \neq 0$, and q has just one zero, a simple zero at $x = 0$; f may be complex.

The equation is transformed by the method of Cherry [Trans. Amer. Math. Soc. **68** (1950), 224–257; MR **11**, 596] into the form $\varepsilon(d^2\eta/du^2) + u\eta = g$, in which η/y, u and g are expressed as power series in ε, the coefficients in which are determinate from recurrence differential relations. By expansion of g in ascending powers of u and use of the method of variation of parameters, solutions are constructed in the form

$$\eta = \rho \sum_{n=0}^{\infty} \frac{e_n(\rho u) g^{(n)}(u)}{\rho^n},$$

where $\rho = \varepsilon^{-1/3}$ and

$e_n(t) =$

$$\frac{1}{n!W}\left[h_2(t)\int_{-\infty_p}^{t} h_1(\tau)(\tau - t)^n d\tau + h_1(t)\int_{t}^{\infty_p} h_2(\tau)(\tau - t)^n d\tau\right],$$

$h_1(t)$, $h_2(t)$ being Airy functions and W their Wronskian. Recurrence differential relations and series expansions are found for the $e_n(t)$.

Applications are made to the deflection of toroidal shells, for which the author claims to have corrected an error in the work of Clark and Reissner [*Advances in applied mechanics*, vol. 2, pp. 93–122, Academic Press, New York, 1951; MR **13**, 885], and to the rotation of thin propellors.

F. W. J. Olver (Teddington)

5785:

Olver, F. W. J. Linear differential equations of the second order with a large parameter. J. Soc. Indust. Appl. Math. **7** (1959), 306–310.

A short survey of the derivation of asymptotic expansions of solutions of a differential equation via those of approximating differential equations of simpler form. The specific reference is to the work of Langer [Trans. Amer. Math. Soc. **33** (1931), 23–64], Olver [Philos. Trans. Roy. Soc. London. Ser. A **249** (1956), 65–97; **250** (1958), 479–517; MR **18**, 38; **20** #1012], and Thorne [Proc. Cambridge Philos. Soc. **53** (1957), 382–398; MR **19**, 272], where the object is to obtain expansions giving uniform approximation over domains which are as large as possible.

T. M. Cherry (Melbourne)

5786a:

Takahashi, Ken-ichi. Eine erweiterte asymptotische Darstellung der Lösung eines Systems von homogenen linearen Differentialgleichungen, welche von zwei Parametern abhängen. Tôhoku Math. J. (2) **11** (1959), 63–97.

5786b:

Takahashi, Ken-ichi. Über eine erweiterte asymptotische Darstellung der Lösung eines Systems von homogenen linearen Differentialgleichungen, welche von zwei Parametern abhängen. J. Fac. Sci. Univ. Tokyo Sect. I **8** (1959), 1–73.

The first of these papers is the main part of the second. The system $dY/dx = \lambda^m \mu^{m'} A(x, \lambda, \mu) Y$, m and m' integers, is considered in the absence of turning points. Possible

configurations of the parameters are divided into three cases. In each case transformation of the system and parameters leads to a system to which classical theory applies. *N. D. Kazarinoff* (Moscow)

5787:

Wasow, Wolfgang. A turning point problem for a system of two linear differential equations. J. Math. and Phys. **38** (1959/60), 257–278.

The method of R. E. Langer for simple differential equations [see e.g., Bull. Amer. Math. Soc. **40** (1934), 545–582; Trans. Amer. Math. Soc. **34** (1932), 447–480; **36** (1934), 90–106; **67** (1949), 461–490; **80** (1955), 93–123; **81** (1956), 394–410; MR **11**, 438; **17**, 365; **18**, 39] seeking asymptotic expansions of solutions uniformly valid in a full neighborhood of a turning point by finding a suitable "related equation" with a known asymptotic theory and subsequently expanding the solutions in terms of solutions of the related equation, does not seem to lend itself to extension to systems of any degree of generality.

The author initiates a rigorous analysis of such turning point problems for systems by considering the rather special case:

(1) $(\varepsilon/i)z_1' = \alpha_1 z_1 + \varepsilon\beta z_2, \quad (\varepsilon/i)z_2' = \varepsilon\bar{\beta} z_1 + \alpha_2 z_2 \;\; ('=d/d\tau),$

arising in the adiabatic theory of Hamiltonian systems, where $\alpha_j = \alpha_j(\tau)$ $(j=1,2)$ are real functions of the real variable τ and $\beta=\beta(\tau)$ may be complex, with $\alpha_j, \beta \in C^\infty$ in an interval $|\tau| \leqq \tau_0$, and where ε is a small parameter, $0<\varepsilon\leqq\varepsilon_0<1$. If it is assumed that $\alpha_1(0)=\alpha_2(0)$ and $[d(\alpha_2-\alpha_1)/d\tau]_{\tau=0}=0$, then (1) has a simple turning point at $\tau=0$. It is also assumed that $\alpha_1(\tau)\neq\alpha_2(\tau)$, $\tau\neq 0$, $|\tau|\leqq\tau_0$; the Hermitian character of (1) is of no importance here.

The approach (capable of generalization) is a technique called "matching" which has previously been employed largely heuristically; we sketch the method briefly. By existing theories of M. Hukuhara [Mem. Fac. Engrg. Kyushu Imp. Univ. **8** (1937), 249–280] and H. L. Turrittin [*Contributions to the theory of nonlinear oscillations*, vol. II, pp. 81–116, Princeton Univ. Press, Princeton, N.J., 1952, MR **14**, 377], one obtains two different types of asymptotic expansions ($\varepsilon\to 0^+$), one valid in an interval bounded away from the turning point by a positive distance (independent of ε), the other in an interval containing the turning point, but shrinking to a point as $\varepsilon\to 0^+$. The fundamental systems of solutions approximated by these two sets of expansions are not necessarily the same. The problem then is how for a given solution the asymptotic expression may be continued from one region to the other. For the case of (1) it is shown that the regions of validity of the two types of expansions are actually larger than the classical theories claim, thus enabling one to effect a matching at a suitably chosen symmetrically located point (depending on ε). The asymptotic nature of the matching matrices depending on ε, but independent of τ, is investigated.

For the purpose of obtaining the two types of expansions, applying the above mentioned known theories, and effecting the matching, it is convenient to change (1) to the form

(2) $$\varepsilon\dot{u} = (A_0+\varepsilon A_1)u \quad (\dot{} = d/dt),$$

$$A_0 = \begin{pmatrix} -it & 0 \\ 0 & it \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & i\gamma(t) \\ i\bar{\gamma}(t) & 0 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where $\gamma(t) = 2\beta(\tau)\dot{t}/[\alpha_2(\tau)-\alpha_1(\tau)]$, by means of a transformation of the variables τ, z_j $(j=1,2)$ which maps the interval $|\tau|\leqq\tau_0$ in a one to one fashion on an interval J of the t axis containing the origin $t=0$, a simple turning point for (2). In the rigorous part of the analysis the class of functions possessing, in a certain sense, an asymptotic development of the form $\sum_{r=r_0}^{\infty} p_r(\log t)t^r$, $t\to 0^+$ —where $p_r(z)$ $(r=r_0,\ r=r_0+1,\ \cdots)$ are polynomials in z, and r_0 is an integer, not necessarily positive—play a key role. Properties of functions in this class are developed.

J. A. Nohel (Atlanta, Ga.)

5788:

Langenhop, C. E. On bounded matrices and kinematic similarity. Trans. Amer. Math. Soc. **97** (1960), 317–326.

Let M_n be the set of all n by n matrices whose entries are complex-valued functions of a real variable t which are continuous and bounded on the whole real line. Let $A(t)$, $B(t)\in M_n$. If there exists a $P(t)\in M_n$ such that $P^{-1}(t)\in M_n$ and $P^{-1}(AP-P')=B$ (here $P'=dP/dt$), then one says A is completely kinematically similar to B (written $A\sim B$). Assume $A(t)\sim B$ where B is a constant matrix with characteristic roots λ_j of multiplicities ρ_j $(j=1,\cdots,q)$. Assume further that $\mathrm{Re}\,(\int_0^t [\mathrm{tr}\,A(s) - \sum_{j=1}^q \rho_j\lambda_j]ds)$ is bounded on $-\infty<t<\infty$. The author then shows that for each j there are ρ_j independent bounded vector functions of t, say $P_1, P_2, \cdots, P_{\rho_j}$, which satisfy the equations:

$$P_1' = (A(t)-\lambda_j I)P_1(t)$$

$$P_s' = (A(t)-\lambda_j I)P_s(t) - P_{s-1} \quad (s=2,\cdots,\rho_j).$$

Furthermore, the condition $\inf|P_1(t)|>0$ is satisfied, and ρ_j is the maximum dimension for any set of functions satisfying the above conditions. The converse of this result is also established under the assumption that $\mathrm{Re}\,(\int_0^t [\mathrm{tr}\,A(s)-\sum_{j=1}^q \rho_j\lambda_j]ds)$ is bounded on $-\infty<t<\infty$. The author then uses these results to generalize an estimate of Markus for the characteristic exponents of the system $x'=A(t)x$ in terms of the maximum and minimum characteristic roots of the matrix $H(t)=[A(t)+A^*(t)]/2$. Here $A^*(t)$ denotes the conjugate transpose of $A(t)$.

J. C. Lillo (W. Lafayette, Ind.)

5789:

Yoshizawa, Taro. Liapunov's function and boundedness of solutions. Funkcial. Ekvac. **2** (1959), 95–142.

This is a comprehensive survey on the application of Lyapunov's second method to the problems of boundedness. Eight types of boundedness (equi-boundedness, ultimate boundedness, etc.) are defined and their mutual relations established in different cases (equations which are linear, periodic, etc.); several examples contribute to clarifying these questions. More than thirty theorems, propositions and corollaries are then proved on the application of Lyapunov functions; many of these results have been published previously by the author, others are at least partially new. *J. L. Massera* (Montevideo)

5790:

Corduneanu, C. Sur l'existence des solutions bornées d'une classe d'équations différentielles du second ordre. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **10**

(1959), 27–34. (Romanian. Russian and French summaries)

Assume $f(x, y, z)$ continuous in the region $x \geq 0$, $|y| \leq Y$, $|z| \leq YM^{1/2}$; $0 < m \leq f_y \leq M$, $|f(x, y, z) - f(x, y, z')| \leq L|z - z'|$, $\sup_{x \geq 0} \int_x^{x+1} |f(t, 0, 0)| dt = N$. Then, if $L < mM^{-1/2}$, $MN + LM^{1/2}Y < mY$, for each sufficiently small y_0 there is one and only one solution of $y'' = f(x, y, y')$ satisfying $y(0) = y_0$, $|y(x)| \leq Y$, $|y'(x)| \leq YM^{1/2}$, $x \geq 0$. There is a similar result for equations defined in $-\infty < x < +\infty$.

J. L. Massera (Montevideo)

5791:

Alekseev, V. M. The asymptotic behaviour of solutions of slightly non-linear systems of ordinary differential equations. Dokl. Akad. Nauk SSSR **134** (1960), 247–250 (Russian); translated as Soviet Math. Dokl. **1**, 1035–1038.

The author states estimates for the growth of a solution of the equation $x' = A(t, x)x + f(t, x)$, where A and f satisfy the standard existence and uniqueness requirements for all $x \in R^n$ and all t in some finite or infinite interval $T \subset R$. These estimates are derived from the following result. Let $x(s)$, $B(s) = A(s, x(s))$ be absolutely continuous in $[t_0, t_1] \subset T$ and such that $\|dB(s)/ds\| \leq \psi(s)$, $\|dx(s)/ds - B(s)x(s)\| \leq \varphi(s)$, where $\varphi(s)$ and $\psi(s)$ are integrable in $[t_0, t_1]$, and suppose that $\|\exp[B(s)\tau]\| \leq \eta(\tau, s)$ for $0 \leq \tau \leq t_1 - s$, where η is bounded. Then $\|x(t)\| \leq g(t)$ for all $t \in [t_0, t_1]$, where g is a solution of the equation

$$g(t) = \eta(t - t_0, t_0)\|x(t_0)\| + \int_{t_0}^{t} \varphi(s)\eta(t-s, s)\, ds + \int_{t_0}^{t} K(t, s)\psi(s)g(s)\, ds$$

with $K(t, s) = \int_s^t \eta(t-u, s)\eta(u-s, s)du$. For the given estimates the integral equation is solved by use of the Laplace transformation.

H. A. Antosiewicz (Los Angeles, Calif.)

5792:

Murthy, Pavman. On the asymptotic behaviour of solutions of the perturbed differential equations. Math. Student **27** (1959), 47–49.

Let $z' = Az + f(z, t)$ and suppose that $y' = Ay$ is restrictively stable and $\|f(\exp(tA)z, t)\| \leq w(\|\exp tA\| \|z\|, t)$ where w is continuous on $R_+ \times R$ and such that the maximal solution $b(t)$ of $x' = cw(\|\exp tA\|x, t)$, $c > 0$, through $(0, x_0)$ is bounded as $t \to \infty$ for every $x_0 \geq 0$. The author shows that then $\|z(t)\| \leq cb(t)$ as $t \to \infty$.

H. A. Antosiewicz (Los Angeles, Calif.)

5793:

Germaidze, V. E. On asymptotic stability of systems with lagging argument. Uspehi Mat. Nauk **14** (1959), no. 4 (88), 149–156. (Russian)

The author considers systems of the form

$$\dot{x}_i = X_i(t, x_1(t - h_{i1}(t)), \cdots, x_n(t - h_{in}(t))) + R_i(t, x_1(t - h_{i1}(t)), \cdots, x_n(t - h_{in}(t)), x_1(t - \eta_{i1}(t)), \cdots, x_n(t - \eta_{in}(t))),$$

where X_i is continuous and lipschitzian for $\|x\| < H$, $t \geq 0$, $X_i(t, 0) \equiv 0$, and where h_{ij}, η_{ij} are piecewise continuous functions with values in $[0, h]$. Using methods similar to those of the reviewer [J. London Math. Soc. **31** (1956), 208–212; MR **18**, 42], he proves that the trivial solution $x(t) \equiv 0$ is asymptotically stable provided the following conditions hold: every solution of the system

$$\dot{x}_i = X_i(t, x_1(t - h_{i1}(t)), \cdots, x_n(t - h_{in}(t)))$$

satisfies for $t \geq t_0$

(1) $\|x(t_0, x_0(t_0 - \theta), t - \tau)\|_\tau \leq B\|x_0(t_0 - \theta)\|_\theta \exp[-\alpha(t - t_0)]$,

and $|R_i| \leq \varphi(t)\|x(t - \tau)\|_\tau$ where, for some $T > 0$ and some $\gamma > 0$ sufficiently small,

$$(2) \qquad \frac{1}{T} \int_{t_0}^{t_0+T} \varphi(t)\, dt \leq \gamma$$

for every $t_0 \geq 0$. Here $\|x(t-\tau)\|_\tau = \sup(\|x(t-\tau)\| : \tau \in [0, 2h])$. If $h_{ij}(t) \equiv 0$, the assertion remains true when (1) is modified to the usual statement of exponential-asymptotic stability and $|R_i(t, y, z)| \leq \varphi(t) \max[|y_i|, |z_i|]$ where $\varphi(t)$ satisfies (2). If $R_i \equiv 0$, the assertion is proved under the assumption that

$$\left| \frac{\partial}{\partial t} X_i(t, x_1(t - h_{i1}(t)), \cdots, x_n(t - h_{in}(t))) \right| \leq \varphi(t)\|y(-\theta)\|_\theta,$$

where $\varphi(t)$ satisfies (2) and $y(-\theta)$ is a particular solution, and that for every $\mu \geq 0$, every solution of the system

$$\dot{x}_i = X_i(\mu, x_1(t - h_{i1}(t)), \cdots, x_n(t - h_{in}(t)))$$

satisfies (1) with α, B independent of μ.

H. A. Antosiewicz (Los Angeles, Calif.)

PARTIAL DIFFERENTIAL EQUATIONS

5794:

Hellwig, Günter. ★Partielle Differentialgleichungen: Eine Einführung. Mathematische Leitfäden. B. G. Teubner Verlagsgesellschaft, Stuttgart, 1960. 246 pp. DM 29.80.

This book provides an excellent introduction to the theory of partial differential equations of the second order. It is intended for the more senior student who has a sound knowledge of the theory of the differential and integral calculus and of ordinary differential equations, and who also knows a little function theory and functional analysis. It is definitely mathematics, and is not a reference book for the man who merely wishes to solve particular equations.

After an introduction in the first chapter to the simplest types of equation (wave equation, potential equation, diffusion equation), the author discusses in Chapter II the normal forms and characteristic manifolds of second order equations and also of systems of first order equations. The third chapter deals with uniqueness theorems with the aid of the maximum-minimum principle and the "energy integral". Existence theorems are discussed in Chapter IV by the methods of classical analysis. The last chapter contains a brief introduction to functional analysis and its applications to the proof of existence theorems.

E. T. Copson (St. Andrews)

5795:

Pawelski, W. Estimation du domaine d'existence de l'intégrale d'un système en involution d'équations aux

dérivées partielles du premier ordre dans le cas de variables complexes. Ann. Polon. Math. **5** (1958), 25–32.

The equations in question form a completely determined compatible analytic system (of Frobenius type) in which the unknown function depends in addition on a number of parameters; the first derivatives of the unknown function with respect to these parameters appear in the equations. The size of the domain of existence turns out to depend on bounds in the complex domain for the first two derivatives of everything in sight.

P. D. Lax (New York)

5796:

Dragan, I. Sur les systèmes de Pfaff associés aux équations aux dérivées partielles du troisième ordre à caractéristiques distinctes. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **10** (1959), 73–94. (Romanian. Russian and French summaries)

The author characterizes those Pfaffian systems associated with a partial differential equation of order three in two independent variables and with distinct characteristics; and he uses his results to determine those partial differential equations of the third order which are integrable by Darboux's method. [See also the author, same Stud. **7** (1956), no. 1, 71–117; MR **20** #1833 for corresponding work on equations of order two.]

A. Erdélyi (Pasadena, Calif.)

5797:

Strang, W. Gilbert. Difference methods for mixed boundary-value problems. Duke Math. J. **27** (1960), 221–231.

Let a bounded domain D^- in a real m-space, with simply-connected interior D and boundary ∂D, be a union of closed unit cubes whose vertices have integral coordinates. Let $C_0^k(D)$ be the space of k-times continuously differentiable functions $g(x)$ in D whose derivatives up to the order k vanish on ∂D. $C_0^k(D)$ is a Banach space with the "k-norm" defined by $\|g\|_k=\sup_{x\in D}|g(x)|+\sum \sup_{x\in D}|D^kg(x)|$, the sum being taken over all derivatives of order k. The problems of linear parabolic and hyperbolic equations with coefficients analytic in D^- are written in matrix notation as $v_t=\sum_i B_i(x)v_{x_i}+C(x)v+F(x,t)$. The author's difference method for the equation is developed with respect to the k-norm: it is proved that (1) the method is "stable" with respect to one k-norm if and only if it is stable with respect to every k-norm, and (2) with respect to any fixed k-norm, the difference approximation is "convergent" if and only if it is "stable".

K. Yosida (Tokyo)

5798:

Tanimura, Masayoshi. On the solution of some mixed boundary problems. IX. Simpler problems of a rectangular domain. Tech Rep. Osaka Univ. **9** (1959), 51–65.

[For Part VIII, see same Rep. **7** (1957), 307–313; MR **20** #2563b.]

Author's summary: "The author's method of solving some mixed boundary value problems by series of repeated contour integrals is extended for more general differential equations on the basis of Sturm-Liouville expansion and applied to some problems of five boundary conditions on the periphery of a rectangular domain."

5799:

Tanimura, Masayoshi. On the solution of some mixed boundary problems. X. The problem of an L-shaped domain. Tech. Rep. Osaka Univ. **10** (1960), 75–84.

Author's summary: "The method of solving differential equations subject to mixed boundary conditions by series of repeated contour integrals was extended to some cases of an L-shaped domain introducing double series of repeated contour integrals."

R. V. Churchill (Ann Arbor, Mich.)

5800:

Kreith, Kurt; Wolf, František. On the effect on the essential spectrum of the change of the basic region. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 312–315.

Let

$$Lu=-\sum_{i,j=1}^n \frac{\partial}{\partial s_j}\left(a_{ij}(s)\frac{\partial u}{\partial s_i}\right)+c(s)u$$

be a singular elliptic operator in a domain G, where

$$\sum_{i,j=1}^n a_{ij}\xi_i\bar{\xi}_j \geq \sigma(s)\sum_{i=1}^n |\xi_i|^2,$$

$\sigma(s)>0$ inside G but is allowed to go to zero on the boundary ∂G of G, and $c(s)$ is bounded below by γ in B. It was proved previously [Wolf, Ann. Mat. Pura. Appl. (4) **49** (1960), 167–179; MR **22** #2783] that under Dirichlet boundary conditions L can be defined so as to become a self-adjoint operator with a spectral decomposition $L=\int_\gamma^\infty \lambda E(d\lambda)$. The essential spectrum $\sigma_e(L)$ consists of those λ for which every neighborhood $(\lambda-\varepsilon,\lambda+\varepsilon)$ corresponds to an infinite dimensional spectral measure $E(\lambda+\varepsilon)-E(\lambda-\varepsilon)$. The present paper is concerned with

$$l=\inf \sigma_e(L)=\sup\{\lambda|\dim E(-\infty,\lambda)<\infty\}.$$

If G_0 is any subdomain of G and l_0 is defined for G_0 in a similar manner, it is proved that $l_0\geq l$. Moreover, if $\partial G_0\cap G$ is bounded away from those points of ∂G where $\sigma(s)=0$, then $\sigma_e(L_0)\subseteq\sigma_e(L)$. An example is given to show that the hypothesis concerning $\partial G_0\cap G$ in the second result cannot be dropped.

M. Schechter (New York)

5801a:

Nečas, Jindřich. Résolution du problème de Dirichlet pour les équations elliptiques aux dérivées partielles du second ordre. Czechoslovak Math. J. **9** (84) (1959), 467–469. (Russian. French summary)

5801b:

Nečas, Jindřich. Sur les solutions des équations elliptiques aux dérivées partielles du second ordre avec intégrale de Dirichlet non-bornée. Czechoslovak Math. J. **10** (85) (1960), 283–289. (Russian. French summary)

Paper (a): The author studies boundary value problems for equations of the form

$$D[u]=\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a^{ij}\frac{\partial u}{\partial x_j}\right)-cu=f$$

for regions Ω and boundary data h of greater generality than have heretofore been discussed. The boundary $\dot{\Omega}$ of Ω is supposed merely to be representable locally by

continuous functions having bounded generalized derivatives of first order. It is assumed that in the closure of Ω, the $a^{ij} \in C^{(1)}$, $c \in C^{(0)}$, and that $c \geqq 0$.

Let $f \in L_2(\Omega)$. Then there is a unique solution $v(x)$ of the equation $D[u]=f$ such that v lies in the closure, in the sense of square integral norm over Ω of the first derivatives, of functions with compact support in Ω. The author states that for this $v(x)$ it is possible to define a generalized conormal derivative $\partial v/\partial \nu$ on $\dot{\Omega}$ such that $\partial v/\partial \nu \in L_2(\dot{\Omega})$. Using this result, the author obtains a generalized solution of $D[u]=0$ with $u=h$ on $\dot{\Omega}$. Here $h \in L_2(\dot{\Omega})$ but is otherwise arbitrary. Details are not included in part a.

Paper (b): Detailed demonstration of the result indicated in (a), plus a concise and clearly presented outline of the abstract (Hilbert space) theory of the equation considered. *R. Finn* (Stanford, Calif.)

5802:

Fiorenza, Renato. Sui problemi di derivata obliqua per le equazioni ellittiche. Ricerche Mat. **8** (1959), 83–110.

Using Miranda's notation [*Equazioni alle derivate parziali di tipo elliptico*, Springer, Berlin, 1955; MR **19**, 421], the result is the following. Given $F(p_{ik}, p_i, u, x)=0$, $x \in T-FT$, $G(p_i, u, x)=0$, $x \in FT$, where F is a bounded domain in S_m of class $A^{(2,\lambda)}$, F is of class C^2 for $x \in T$; G is of class C^2 with Lipschitzian second derivatives for $x \in FT$; the quadratic form $\sum_{i,k}{}^m F_{p_{ik}}\lambda_i\lambda_k$ is positive definite for $x \in T$; if $\gamma_i(x)$ are the direction cosines of the exterior normal to FT, then $\sum_i{}^m G_{p_i}\gamma_i \geqq h>0$ for $x \in FT$; ψ is of class $C^{(1,\lambda)}$ on FT. It is assumed that: (1) for every solution of class $C^{(2,\lambda)}$ of $F=0$, the corresponding variational problem is unlimitedly solvable; (2) as ψ varies over a family of functions for which, if Ψ is the restriction of ψ to FT, $\Psi+\Psi_{1,\lambda}$ is uniformly bounded; (3) there is one and only one solution for one particular such ψ. Under these conditions, there is one and only one solution of the problem.

If F is semilinear, only boundedness of the first derivatives is required in (2). The methods are extensions of the well-known ones used by Caccioppoli, Schauder, Bers and Nieremberg. *G.-C. Rota* (Cambridge, Mass.)

5803:

Redheffer, Raymond M. Maximum principles and duality. Monatsh. Math. **62** (1958), 56–75.

Let u and v be real functions of class C^2 with partial derivatives u_{ij}, etc., defined on an x-region $R \subset E^n$. The notation $u \leqq o(v)$ at the boundary is used to mean that there is an increasing sequence of compact subsets $R_1 \subset R_2 \subset \cdots$ of R, $R=\bigcup_{k=1}^{\infty} R_k$, and a sequence $\varepsilon_k \to 0$ such that $v \geqq -k\varepsilon_k$ and $u \leqq k^{-1}|v|+k^{-1}$ hold on the boundary of R_k. The author proves a number of maximum principles for u and v, i.e., assertions that if u and v satisfy certain conditions, then $u \leqq o(v)$ [or $u-m \leqq o(v)$, $m=\text{const}$] at the boundary implies $u \leqq 0$ [or $u \leqq m$] in R. If $v \equiv 1$, such assertions are (weak) maximum principles in the usual sense. A typical example is theorem I: Let $f(u_{ij})=0$ and let the matrix (v_{ij}) be non-positive semi-definite in R. Suppose that $f(c_{ij}) \neq 0$ for all negative definite (c_{ij}). Then, if $u-m \leqq o(v)$ at the boundary, $u \leqq m$ in R. The main purpose of the paper is to give a unified treatment of various phenomena associated with partial differential equations, e.g., theorems of the Phragmén-Lindelöf and Liouville types on the growth of solutions and theorems of the Sturm type on the zeros of solutions. This is done by showing that these phenomena are consequences of maximum principles for u and v.

{A minor slip occurs in the proofs of theorems V and VI where the properties required of the set S are not consistent, but suitable properties are easily found. The first sentence in the next to the last paragraph of the proof of theorem VI should read: "In the linear case, the part of the O term in (21) coming from the Lipschitz condition on a_{ij} is missing."} *R. Sacksteder* (New York)

5804:

Kapilevič, M. B. The theory of linear differential equations with two perpendicular lines of parabolicity. Dokl. Akad. Nauk SSSR **125** (1959), 251–254. (Russian)

The author considers solutions of the differential equation

$$\Delta u+\{m(r)/x\}u_x+\{n(r)/y\}u_y+F(r)u = 0$$

in $\Omega=(x \geqq 0, y \geqq 0)$, where $r=(x^2+y^2)^{1/2}$, $m(r)$ and $n(r)$ are positive entire functions, $0<m(0)$, $n(0)<1$ and $F(r)$ has a simple pole at $r=0$. The solutions considered satisfy either $u_\xi(0, y)=u_\eta(x, 0)=0$ and $u(0, 0)\neq 0$, or $u(0, y)=u(x, 0)=0$, $u_{\xi\eta}(0, 0)\neq 0$, where $\xi=\{x/[1-m(0)]\}^{m(0)}$ and $\eta=\{y/[1-n(0)]\}^{1-n(0)}$. The author derives formulas for certain mean values of such solutions on $\{x^2+y^2=r^2\} \cap \Omega$. The results are extensions of his earlier results for constant m and n [same Dokl. **125** (1959), 719–722; MR **21** #4289]. *D. G. Aronson* (Minneapolis, Minn.)

5805:

Moore, Robert H. On approximate solutions of non-linear hyperbolic partial differential equations. Arch. Rational Mech. Anal. **6**, 75–88 (1960).

The Cauchy problem for $u_{xy}=f(x, y, u, u_x, u_y)$ (f Lipschitz continuous in u_x, u_y) is solved by a finite difference method analogous to Peano's method for ordinary differential equations. *P. Ungar* (New York)

5806:

Sestini, Giorgio. Ancora su di un teorema di unicità in problemi unidimensionali analoghi a quello di Stefan. Boll. Un. Mat. Ital. (3) **14** (1959), 373–375. (English summary)

The author claims to have corrected his proof in a previous paper on this subject [same Boll. (3) **12** (1957), 516–519; MR **20** #1086]. *C. R. DePrima* (Pasadena, Calif.)

5807:

Pucci, Carlo. Alcune limitazioni per le soluzioni di equazioni paraboliche. Ann. Mat. Pura Appl. (4) **48** (1959), 161–172. (English summary)

The author studies the dependence of positive solutions of the parabolic equation $u_t=Lu$ upon Cauchy data specified on a set of the form $F=B\times(0<t<l)$, where B is a subset of the hyperplane $x_n=0$. Let

$$(1) \qquad Lu = \sum a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum b_i\frac{\partial u}{\partial x_i}+cu,$$

where a_{ij}, b_i, c are Hölder continuous, $c \leqq 0$, and $\{a_{ij}\}$ is uniformly positive definite on a space-time cylinder T

containing F as a portion of its boundary. Let Γ' be the class of functions that are continuous on the closure of T, have continuous second spatial derivatives and a continuous time derivative in T, and have a normal derivative on F. He generalizes the Cauchy problem to consider solutions $u \in \Gamma'$ of

$$(2)\quad \begin{aligned} u_t &= Lu \text{ in } T,\\ |u-\phi| &\leq \varepsilon(\|\phi\|_F+\|\psi\|_F) \text{ on } F,\\ |u_n-\psi| &\leq \varepsilon(\|\phi\|_F+\|\psi\|_F) \text{ on } F,\\ u &\geq 0 \text{ on } T. \end{aligned}$$

The norm is the least upper bound on F. Denote by U_ε the packet of solutions of (2), and let

$$\delta_\varepsilon(D) = \sup_{(x,t)\in D;\, u,v\in U_\varepsilon} |u(x,t)-v(x,t)|$$

denote the diameter of U_ε with respect to $D \subset T \cup F$. Clearly, $\delta_\varepsilon(D)$ is a measure of the dependence of the solution of the differential equation on the Cauchy data. Assume that L is such that the solution of the Cauchy problem for F is unique. The author proves that there exists $w(\varepsilon)$ for positive ε such that $\lim_{\varepsilon\to 0} w(\varepsilon)=0$, $\delta_\varepsilon(D) \leq (\|\phi\|_F+\|\psi\|_F)w(\varepsilon)$, for any compact D. Thus, the non-negativity of u converts an otherwise improperly posed problem into a properly posed problem. The proof is based on two a priori bounds, the second of which states that $u(x,t) \leq k_1(\|u\|_F+\|u_n\|_F)$ in D if $u \geq 0$, $u \in \Gamma'$, and $u_t = Lu$. The proofs are clearly presented.

J. Douglas, Jr. (Houston, Tex.)

5808:

Yamabe, Hidehiko. Kernel functions of diffusion equations. II. Osaka Math. J. 11 (1959), 1–6.

In a previous paper [same J. 9 (1957), 201–214; MR 21 #2813] the author has given a method for constructing the kernel function $K(x, y, t)$ for the diffusion equation subject to zero boundary condition over a bounded open set D with smooth boundary. In this paper the author observes (a) that the integral of K with respect to t is the Green's function of the Laplace equation over D, (b) that one can construct a kernel function for any bounded open set D as the monotone limit of the kernel functions of a monotone sequence of open subsets of D with smooth boundaries. The integral of this kernel function is called the generalized Green's function $G(x, y)$ of D. The main result of this paper is: Let y_m be a sequence of points of D tending to a boundary point such that $G(x, y_m)$ tends to zero for every point x of D; then also $K(x, y_m, s)$ tends to zero for any positive s and every x in D. The proof is based on the reproducing property of K, the equicontinuity of K on compact subsets of D and an ingenious L_2 inequality proved with the aid of an eigenfunction expansion.

P. D. Lax (New York)

5809:

Krzyzanski, M.; Szybiak, A. Construction et étude de la solution fondamentale de l'équation lineaire parabolique aux coefficients non bornés. Symposium on the numerical treatment of partial differential equations with real characteristics: Proceedings of the Rome Symposium (28–29–30 January 1959) organized by the Provisional International Computation Centre, pp. 32–36. Libreria Eredi Virgilio Veschi, Rome, 1959. xii+158 pp.

The authors consider

$$(1)\quad \mathscr{F}u \equiv \sum_{i,j=1}^{m} a_{ij}(x,t)u_{,ij}+\sum_{j=1}^{m} b_j(x,t)u_{,j}+c(x,t)u-u_{,t} = 0$$

for $(x, t) \in E^m \times (0, T_0) \equiv R$, where the a_{ij}, b_j are bounded and belong to $C^3(R)$, and $\mathscr{F}$ is uniformly parabolic in R. It is assumed that c is continuous, satisfies a Lipschitz condition in x, and $|c| \leq \alpha|x|^2+\beta$ for some positive constants α and β. The authors sketch a proof of the existence of a fundamental solution of (1). The result is a special case of the results of Èidel'man [Mat. Sb. (N.S.) **38 (80)** (1956), 51–92; MR **17**, 857]. In addition, various properties of the fundamental solution are discussed; particularly its behavior as $t\to+\infty$ in case the a_{ij} are constant and $b_j=0$.

D. G. Aronson (Minneapolis, Minn.)

5810:

Malgrange, B. Sur les systèmes d'équations elliptiques. La théorie des équations aux dérivées partielles. Nancy, 9–15 avril 1956, pp. 139–143. Colloques Internationaux du Centre National de la Recherche Scientifiques, LXXI. Centre National de la Recherche Scientifique, Paris, 1956. 187 pp. 1500 francs.

The following theorems are stated: (I) Let D be an elliptic operator with analytic coefficients, G a domain in Euclidean space whose complement doesn't contain compact components. Then every solution of $Df=0$ in G can be approximated uniformly on compact subsets by solutions f_i of $Df_i=0$ in the whole space. (II) Let g be a function defined in the whole space, locally smooth but subject to no condition at infinity; then $Df=g$ has a solution in the whole space. The proofs, sketched briefly, run as follows: (I) is proved by duality, the main ingredients being the alternative for the operator D, Weyl's lemma and the unique continuation theorem for analytic elliptic equations. The proof of (II) is based on the Mittag-Leffler trick for constructing meromorphic functions with prescribed poles; theorem (I) is used.

Detailed exposition of the proofs is given in the author's paper in Ann. Inst. Fourier. Grenoble **6** (1955/56), 271–355 [MR **19**, 280].

P. D. Lax (New York)

5811:

Lavruk, B. R. On the index of a certain operator of the boundary problem for an elliptic system of linear differential equations of the second order. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 287–290. (Russian)

The writer states some results concerning the following boundary value problem: Let

$$A\left(x, \frac{\partial}{\partial x}\right) = \sum_{i,j} A_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}+\sum_i A_i(x)\frac{\partial}{\partial x_i}+A(x),$$

be an elliptic system of rank p in a domain D of Euclidean n-space, with boundary S. Let

$$B\left(y, \frac{\partial}{\partial x}\right) = \sum_i B_i(y)\frac{\partial}{\partial x}+B(y)$$

be a boundary operator of rank p (i.e., the $B_i(y)$ and $B(y)$ are matrices of rank p) defined and regular on S. If $n_i(y)$ are the components of the normal to S at y, it is assumed that $\operatorname{Det}(\sum_i B_i(y)n_i(y)) \neq 0$ for y in S, while

$$\sum_i B_i(y)n_i(y) = \sum_{i,j} A_{ij}(y)n_i(y)n_j(y).$$

Consider the boundary-value problem $A(x, \partial/\partial x)u = F$, x in D; $B(y, \partial/\partial x)u \to f(y)$ as $x \to y$ on S. Let A^* and B^* be the operators of the dual problem. The writer states that using results of Ya. B. Lopatin'skiĭ [Dopovidi Akad. Nauk. URSR 1956, 107–112, 211–213; MR **20** #1074; **19**, 555], he can show that the index of the boundary value problem = {number of null solutions of (A, B) − number of null solutions of (A^*, B^*)} does not depend on the lower order terms $A_i(x)$, $A(x)$, and $B(y)$.

F. Browder (New Haven, Conn.)

5812:

Lavruk, B. R. The index of a boundary problem operator for an elliptic system of second order linear differential equations, as dependent on the highest coefficients. Dokl. Akad. Nauk SSSR **121** (1958), 970–972. (Russian)

This paper is a continuation of the author's work in same Dokl. **111** (1956), 23–25, 287–290 [MR **19**, 554; and preceding review]. It is shown here that the index of the operators $A(x, \partial/\partial x)$, $B(y, \partial/\partial x)$ does not change under arbitrary variation of the leading coefficients if none of the following conditions are violated: the condition of ellipticity for $A(x, \partial/\partial x)$; that $\det \sum_{i=1}^{n} B_i(y)\nu_i(y) \neq 0$ $(y \in S)$, where $\nu(y) = (\nu_1(y), \cdots, \nu_n(y))$ is the unit interior normal to the boundary S of the domain V in which the boundary value problem is being considered; conditions permitting reduction of the boundary value problem for the pair A, B as well as A^*, B^* to regular integral equations.

A. N. Milgram (Berkeley, Calif.)

5813:

Ahlfors, Lars; Bers, Lipman. Riemann's mapping theorem for variable metrics. Ann. of Math. (2) **72** (1960), 385–404.

In this paper, it is proved that if $\mu(z)$ is any bounded measurable function with $|\mu(z)| \leq k < 1$, there is a unique μ-conformal homeomorphism w^μ of the whole plane into itself for which $w^\mu(0) = 0$, $w^\mu(1) = 1$, and $w^\mu(\infty) = \infty$, and a unique μ-conformal homeomorphism W^μ of the unit disk into itself for which $W^\mu(0) = 0$ and $W^\mu(1) = 1$; a function $f(z)$ is μ-conformal if f is continuous and has generalized derivatives locally in L_2 which satisfy $f_{\bar z} = \mu f_z$ almost everywhere.

The principal aim of the paper is to examine the dependence of w^μ and W^μ on μ. For example, the following results are proved: (1) if $\mu_n \to \mu$ almost everywhere $(|\mu_n(z)| < k < 1)$, then $\|w^{\mu_n} - w^\mu\|_{B(R,p)} \to 0$ for each R; (2) if μ depends holomorphically on complex parameters $(\gamma_1, \cdots, \gamma_n)$ as an element of L_∞, then w^μ is a holomorphic function of these parameters, as an element of $B(R, p)$; here $B(R, p)$ is the Banach space of w in which

$$\|w\|_{B(R,p)} = \|w\|_{L(R,p)} + \sup_{|z|, |\zeta| \leq R} |\zeta - z|^{-\lambda} |w(\zeta) - w(z)|,$$
$$\lambda = 1 - 2/p,\ p > 2.$$

More detailed results are obtained as are corresponding results concerning W^μ. The paper is largely self-contained and the presentation is excellent.

C. B. Morrey, Jr. (Berkeley, Calif.)

5814a:

Jaenicke, Joachim. Nullstellenvorgaben für Lösungen linearer Randwertprobleme elliptischer Differentialgleichungssysteme. Math. Nachr. **18** (1958), 106–119.

5814b:

Jaenicke, Joachim. ★Über lineare Randwertprobleme der Systeme von zwei linearen partiellen Differentialgleichungen erster Ordnung vom elliptischen Typus. Dissertation, Technische Universität Berlin, Berlin, 1957. ii+56 pp.

These papers generalize results of Haack, Hellwig, Vekua and others concerning boundary value problems of the form

$$\alpha u + \beta v = f(s) \tag{1}$$

for linear elliptic systems of the form

$$\begin{aligned} u_x - v_y &= au + bv + c \\ u_y + v_x &= du + ev + g. \end{aligned} \tag{2}$$

It is assumed that (x, y) ranges over a simply connected closed region F with boundary curve R having a Hölder continuously turning tangent. The coefficients a-g are assumed bounded and Hölder continuous except for a finite number of points. If the vector $\{\beta_0(s), \alpha_0(s)\}$ is $\{\beta, \alpha\}$, normalized to have unit length, it is required to have only a finite number of discontinuities and to have uniformly Hölder continuous derivatives on the intervals between successive discontinuities. If we choose $w(s)$ to be continuous on each of these intervals so that

$$\beta_0(s) = \cos w(s), \quad \alpha_0(s) = \sin w(s),$$

then the characteristic of the vector $\{\beta, \alpha\}$ is defined as the sum of the variations of w over these intervals.

The results of the second paper are as follows: The boundary value problem (1) for the homogeneous equations (2) has a solution with preassigned poles and zeros on the closed region F (some of these on $R = \partial F$ may be of non-integral order) provided that the sum of the characteristics of the boundary values of the functions $(z - z_j)^{\nu_j}$ (a pole or zero at z_j) equals that of the vector $\{\beta, \alpha\}$, the discontinuities of $\{\beta_0, \alpha_0\}$ are cancelled by appropriate zeros or poles on R, and f satisfies a continuity condition too long to state here.

In the first paper, the restrictions or R and the coefficients are less restrictive than those given above but the results are less complete.

C. B. Morrey, Jr. (Berkeley, Calif.)

5815:

D'yačenko, V. F. The Cauchy problem for quasilinear systems. Dokl. Akad. Nauk SSSR **136** (1961), 16–17 (Russian); translated as Soviet Math. Dokl. **2**, 7–8.

The author points out that under the usual general definition of a solution, Cauchy's problem does not have a unique solution for a certain hyperbolic system of quasi-linear equations. The object of this note is to show that even under another definition, proposed by I. M. Gel'fand, Cauchy's problem still need not have a unique solution. This object is accomplished.

H. P. Thielman (Oxnard, Calif.)

5816:

Ludwig, Donald. Exact and asymptotic solutions of the Cauchy problem. Comm. Pure Appl. Math. **13** (1960), 473–508.

The author considers hyperbolic systems of differential equations of any order, and constructs "generalized pro-

gressing wave solutions". These are formal solutions $u(t,x)=\sum_{j=0}^{\infty} f_j[\phi(t,x)]a^j(t,x)$ in which the "wave form" $f_0(s)$ is an arbitrary function or distribution and $f_j(s)=df_{j-1}(s)/ds$. The phase function $\phi(t,x)$ and coefficients $a^j(t,x)$ are determined so that, whatever may be the wave form, the segments $\sum_{j=0}^{N}$ of u are exact solutions. By the use of progressing waves, a Cauchy problem involving singular data can be reduced to one with arbitrarily smooth data. Under certain conditions the expansion for u is actually convergent—independent of the wave form—so that a "Riemann function" may be constructed as in the classical method of Hadamard.

R. W. McKelvey (Los Angeles, Calif.)

5817:

Volkov, D. M. An analog of Lyapunov's second method in non-linear boundary problems for hyperbolic equations. Leningrad. Gos. Univ. Uč. Zap. Ser. Mat.Nauk **33** (1958), 90–95. (Russian)

Suppose all functions entering into a given non-linear hyperbolic equation, as well as the initial and boundary data, belong to a certain class of functions ω_k. Let T_k denote the corresponding set of solutions, and suppose that T_k is non-empty and contains the trivial solution $u\equiv 0$. The author introduces for this trivial solution a notion of stability with respect to ω_k which involves certain non-linear operators on T_k that are required to have properties similar to those of Liapunov functions.

H. A. Antosiewicz (Los Angeles, Calif.)

5818:

Milicer-Grużewska, H. Les propriétés probabilistes de la solution d'un système parabolique d'équations. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 721–728. (Russian summary, unbound insert)

Given a system of N second order linear parabolic partial differential equations with the fundamental solutions $(\Gamma_{i1},\cdots,\Gamma_{iN})$ $(i=1,\cdots,N)$, the author shows that under suitable conditions on the coefficients of the system, the functions

$$P_i(X,t;Y,\tau)=\sum_{j=1}^{N}\Gamma_{ij}(X,t;Y,\tau) \quad (i=1,\cdots,N;\ t<\tau)$$

are transition probability densities. Here X and Y are points in Euclidean n-space. It is also shown that for each i, P_i is the fundamental solution of a single second order linear parabolic partial differential equation.

J. Elliott (New York)

5819:

Milicer-Grużewska, H. Le second théorème d'unicité de solution d'un système parabolique d'équations linéaires avec les coefficients höldériens. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 719–720. (Russian summary, unbound insert)

Announcement of results proved in the paper whose review follows.

J. Elliott (New York)

5820:

Milicer-Grużewska, Halina. Le second théorème d'unicité de solution d'un système parabolique d'équation linéaires avec les coefficients höldériens. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8) **6** (1960), 17–29.

The author proves uniqueness of solution to the Cauchy problem for a system of linear parabolic partial differential equations in a class Z_I' of solutions satisfying the following condition: the spatial derivatives of order not exceeding that of the system, as well as the first time derivative, are continuous in $E^n\times(0,T)$ and are $O(t^{-\mu})$ for some $0\leq\mu<1$. A uniqueness theorem for a different class Z_I was given by the author in Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 593–599 [MR **22** #3886]. The class Z_I contains neither the constants nor the generalized Poisson-Weierstrass integral, whereas Z_I' contains both. However, Z_I is not contained in Z_I'.

J. Elliott (New York)

5821:

Thomée, Vidar. Locally cogent boundary operators. Math. Scand. **7** (1959), 5–32.

Let L be a constant-coefficient partial differential operator in Euclidean space R^{m+1} with points $(t,x_1,\cdots,x_m)$. Let V be a region in R^{m+1} contained in the half-space $t>0$ and having a portion S_0 of its boundary contained in the hyperplane $t=0$. The author considers necessary and sufficient conditions for the inequality

$$(1)\qquad \|u\|_V\leq C\|Lu\|_V$$

to hold for functions u vanishing outside compact subsets of $V\cup S_0$ and having their first $\sigma-1$ derivatives vanishing on S_0, where $\|w\|_V$ is the $L^2(V)$ norm of w. If $L(\xi,\tau)$ is the characteristic polynomial of L, with τ and $\xi=(\xi_1,\cdots,\xi_m)$ corresponding to t and $(x_1,\cdots,x_m)$, respectively, let $\tau_i(\xi)$, $i=1,\cdots,n$, be the zeros of $L(\xi,\tau)$ considered as a polynomial in τ for fixed $\xi\neq 0$. If $p(\xi)$ is the number of such roots with positive imaginary parts, it is shown that (1) holds if, and only if, $p(\xi)\leq\sigma$ for all $\xi\neq 0$. Moreover, in such a case, the following estimate holds:

$$\|u\|_V\leq C(\|Lu\|_V+\sum_{k=1}^{\sigma}\|D_t^{k-1}u\|_{S_0,\sigma-k}),$$

where $\|w\|_{S_0,p}$ is the sum of the $L^2(S_0)$ norms of all tangential derivatives of w up to order p. Generalizations are obtained for two-dimensional operators having simple characteristics.

M. Schechter (New York)

5822:

Goldhagen, E. Sur quelques systèmes d'équations aux dérivées partielles du second ordre. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **10** (1959), 249–268. (Romanian. Russian and French summaries)

Continuing earlier investigations [Akad. R. P. Romîne. Bul. Şti. Sect. Şti. Mat. Fiz. **7** (1955), 623–644; same Stud. **7** (1956), no. 1, 51–70; **8** (1957), no. 1, 75–106; MR **17**, 490; **20** #1104] the author now considers a system of two partial differential equations of the second order in three independent variables x_1, x_2, and x_3, assuming that this system possesses a solution depending on arbitrary functions on x_1 and x_2, and arbitrary functions of x_3. He reduces such systems to a number of standard forms and shows that the explicit solutions are "general solutions" of the equations they satisfy.

A. Erdélyi (Pasadena, Calif.)

5823:

Prodi, Giovanni. Qualche risultato riguardo alle equazioni di Navier-Stokes nel caso bidimensionale. Rend. Sem. Mat. Univ. Padova **30** (1960), 1–15.

O. A. Ladyženskaya [Dokl. Akad. Nauk SSSR **123** (1958), 427–429; MR **21** #7675] showed the existence and uniqueness for all time of the mixed problem for the Navier-Stokes equations in the two-dimensional case, based on the inequality (1) $|u|^2_{L^4(\Omega)} \leq 2^{1/2}|u|\ \|u\|$ (where the left-hand side is the L^4 norm of the vector u, $|u|$ is its L^2 norm, and $\|u\|$ is its Dirichlet norm, and Ω is any open subset of the plane). J. L. Lions and the author [C. R. Acad. Sci. Paris **248** (1959), 3519–3521; MR **21** #7676] gave a new treatment of this problem, and the present paper presents some further results. The author considers a class of weak solutions of the Navier-Stokes equations in Ω and shows that (possibly after a change on a set of t's of Lebesgue measure 0) they are strongly continuous in t (theorem 1). The proof is based on (1) and a bound for the integral of $\|u(t)\|^2$ (lemma 3). The solution depends continuously (strong topology) on the initial data. Theorem 2 asserts that if Ω is bounded the solution depends continuously in the weak topology on the initial data. If the known term in the equations is periodic in t, then there exists at least one periodic weak solution (theorem 3). The proof of this uses the estimate of lemma 3 and the Tychonov fixed-point theorem.

E. Nelson (Princeton, N.J.)

5824:

Lions, J. L. Sur la régularité et l'unicité des solutions turbulentes des equations de Navier Stokes. Rend. Sem. Mat. Univ. Padova **30** (1960), 16–23.

The author gives a simpler proof of theorems 1 and 2 of the paper reviewed above, a proof which includes the uniqueness result. The condition that Ω be bounded in theorem 2 is removed. *E. Nelson* (Princeton, N.J.)

5825:

Lions, Jacques-Louis. Quelques résultats d'existence dans des équations aux dérivées partielles non linéaires. Bull. Soc. Math. France **87** (1959), 245–273.

A very general estimate for the fractional derivative of order $<\frac{1}{4}$ of certain non-linear equations having a finite degree of non-linearity is derived, using the Fourier transform in the time variable. For each real t, let $u_1, \cdots, u_m, v \to b(t; u_1, \cdots, u_m, v)$ be an $(m+1)$-linear form on a Hilbert space V, V contained (with a finer topology) in a Banach space E. Suppose that for all $u_i, v \in V$, the function $t \to b(t; u_1, \cdots, u_m, v)$ is continuous on R. Suppose also that there is a constant β such that

$$|b(t; u_1, \cdots, u_m, v)| \leq \beta \|u_1\|_E \cdots \|u_m\|_E |||v|||,$$

where $\|w\|_E$ is the norm in E and $|||w||| = (((w, w)))^{1/2}$ is the norm in V. Let V be contained (with a finer topology) in a Hilbert space H, with norm denoted by $|w| = (w, w)^{1/2}$. Let u be in $L^2(-\infty, +\infty; V) \cap L^m(-\infty, +\infty; E)$ and be 0 for $t<0$. Let $J_u = \int_0^\infty |||u(t)|||^2 dt$ and $L_u = \int_0^\infty \|u(t)\|_E{}^m dt$. Suppose that for all v in V,

$$(1)\quad D_t(u(t), v) + b(t; u(t), u(t), \cdots, u(t), v) = (((f(t), v))) + (u_0, v)\delta,$$

where $D_t = d/dt$ is calculated in the sense of distributions on R, δ is the Dirac measure at 0, f is given in $L^2(-\infty, \infty; V)$ and is 0 for $t<0$, and u_0 is given in H. Then for any γ with $0<\gamma<\frac{1}{4}$ the fractional derivative $D^\gamma u$ satisfies $\int |D^\gamma u|^2 dt \leq c(\gamma)(J_u + (K_u + |u_0|)J_u{}^{1/2} + \int |||f(t)|||^2 dt)$, where $c(\gamma)$ is a constant depending only on γ.

The Navier-Stokes equations may be written in the form (1) with $b(u, v, w) = \sum_{i,k} \int_\Omega u_k(D_k v_i) w_i dx$, where Ω is an open set in R^n. If $n \leq 4$ the Sobolev theorems imply that b is continuous on $V \times V \times V$, where V is the space of divergence free vector fields on Ω, each of whose components together with its first derivatives is in $L^2(\Omega)$, and which vanish on the boundary in a weak sense. Using the estimate on fractional derivatives and a compactness argument, the author shows, for $n \leq 4$, the existence of a weak solution u of the Navier-Stokes equations with $\int_0^\infty |D^\gamma u(t)|^2 dt < \infty$. The finiteness of this integral is the main new fact not contained in the results of J. Leray [Acta Math. **63** (1934), 193–248] and E. Hopf [Math. Nachr. **4** (1951), 213–231; MR **14**, 327]. For $n=2$ this enables the author to prove uniqueness (see the paper reviewed above). He considers briefly the Navier-Stokes equations perturbed (or smoothed) by the addition of a term $(-1)^m \varepsilon \Delta^m u$, and states that existence holds for $n \leq 2(3m-1)$, uniqueness for $n \leq 2m$; he also applies his method to two equations with non-linearities of degree three.

A brief account of the results appeared in C. R. Acad. Sci. Paris **248** (1959), 2847–2849 [MR **21** #3680].

E. Nelson (Princeton, N.J.)

5826a:

Zlámal, Miloš. Über das gemischte Problem für eine hyperbolische Gleichung mit kleinem Parameter. Czechoslovak Math. J. **9 (84)** (1959), 218–242. (Russian. German summary)

5826b:

Zlámal, Miloš. Sur l'équation des télégraphistes avec un petit paramètre. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **27** (1959), 324–332.

Consider the hyperbolic partial differential equation

$$(1)\quad \varepsilon\alpha(t)u_{tt} + \beta(t)u_t - a(x)u_{xx} = F(x, t),$$

where $\varepsilon > 0$ is a small parameter. For $\varepsilon = 0$, (1) reduces to the parabolic equation

$$(2)\quad \beta(t)U_t - a(x)U_{xx} = F(x, t).$$

In these two papers the author studies the relationship between solutions of the mixed and pure initial value problems for (1) and (2). Specifically let u be the solution of (1) which satisfies $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$, $u(0, t) = u(l, t) = 0$, and let U be the solution of (2) which satisfies $U(x, 0) = f(x)$, $U(0, t) = U(l, t) = 0$. In the first paper the author shows in case $F \equiv 0$ that $u = U + O(\varepsilon)$, $u_x = U_x + O(\varepsilon)$ and

$$u_t = U_t + \{g(x) - U_t(x, 0)\} \exp\left\{-\int_0^t \beta(s)/\alpha(s)\, ds/\varepsilon\right\} + O(\varepsilon^{3/4}).$$

Similar estimates hold for $F \not\equiv 0$ with smaller powers of ε in the O-terms. It is assumed that $\alpha, \beta, a > 0$, $d(\alpha/\beta)/dt \geq 0$, $f \in C^5$, $g \in C^3$ and $f(0) = f(l) = f''(0) = f''(l) = g(0) = g(l)$. To prove this result the author solves (1) by separation of variables and studies the behavior of resulting series..

In the second paper the author considers (1) and (2) for $\alpha \equiv \beta \equiv a \equiv 1$ and u_{xx} replaced by the n-dimensional

Laplacian, where $u(x, 0)=f(x)$, $u_t(x, 0)=g(x)$ and $U(x, 0)=f(x)$ for all $x \in E^n$. A result similar to the one quoted above is proved, in this case by Fourier transform methods. Here it is assumed that $f \in C^{5+n}$ and $g \in C^{3+n}$ have compact support. {Reviewer's comment. In his *Lectures on Cauchy's problem in linear partial differential equations* [Dover, New York, 1953; MR **14**, 474; pp. 103–104] Hadamard shows that the fundamental solution of the heat equation can be obtained as the limit of the Riemann function for the telegrapher's equation with a small parameter.} *D. G. Aronson* (Minneapolis, Minn.)

POTENTIAL THEORY

See also 5740a–c.

5827:

Fuglede, Bent. The symmetric normal derivative of a subharmonic function. Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 90–101. Mercators Tryckeri Helsinki, 1958. 209 pp. (1 plate)

Let Ω be a plane region, let G be a bounded subregion whose boundary Γ consists of a finite number of C'' arcs in Ω, and let ν be the unit inward normal to Γ. For u a function on Ω, let $L_r u$ and $A_r u$ be the circumferential and areal means, respectively, and let $\delta u/\delta\nu$ denote the symmetric normal derivative

$$(1) \qquad \frac{\delta u}{\delta \nu} = \lim_{h\to 0+} \frac{u(x+h\nu)-u(x-h\nu)}{2h}.$$

Three main theorems are presented. Theorem 1: If u is subharmonic on Ω, then $\delta u/\delta\nu$ exists a.e. on Γ. In fact, the limit in (1) exists in the mean over Γ, and

$$\int_\Gamma \frac{\delta u}{\delta \nu}\, ds = 2\pi\mu(G)+\pi\mu(\Gamma),$$

where μ is the mass distribution for u. Theorem 2: For u to be subharmonic on Ω it is necessary and sufficient that (A) $u=\lim_{r\to 0} A_r u$ on Ω and (B) for all admissible Γ, $\int_\Gamma \delta u/\delta\nu\, ds \geqq 0$. Theorem 3: For u to be quasi-subharmonic on Ω it is necessary and sufficient that (a) $u=\lim_{r\to 0} A_r u$ quasi-everywhere on Ω and (b) for all admissible Γ, (1) holds in the mean over Γ and $\int_\Gamma \delta u/\delta\nu\, ds$ determines a set function finite on compact sets.

Although a proof of theorem 1 is given here only for polygonal Γ, the necessity in theorem 2 depends on the general case of theorem 1, and the proof of theorem 3 uses theorem 1 for circular Γ. The key step in proving theorem 2 is to show that condition (B) forces $L_r u$ to be essentially monotone increasing in r. This clearly makes $A_r u$ increasing in r, so that u is almost subharmonic and therefore (by condition (A)) subharmonic. (The author gives an alternative argument independent of the theory of almost subharmonic functions.) In connection with theorem 3 it should be noted that similar problems have been studied, under more restrictive hypotheses, by Myškis [Izv. Akad. Nauk SSSR. Ser. Mat. **17** (1953), 13–30; MR **15**, 424] (attention was called to a C' counterpart of theorem 3 in this review). Theorems 2 and 3 extend in an evident way to n-dimensional space. The same is asserted for theorem 1, but it is not clear to the reviewer how the 2-dimensional case can be adapted to n dimensions here. *M. G. Arsove* (Seattle, Wash.)

5828:

Günzler, Hans. Eine Charakterisierung harmonischer Funktionen. Math. Ann. **141**, 68–86 (1960).

Let $f=f(x)$ be Lebesgue integrable on each compact set in R^n, Euclidean n-space, and let $U=U(x, t)$ be the sphere with center at x and radius $t>0$. Next let $O_t f=O_t f(x)$ denote the mean value of f taken over the surface of U and $V_t f=V_t f(x)$ the mean value taken over the whole of U. The author first proves some theorems on the behaviour of $O_t f$ and $V_t f$ as $t\to\infty$. For example, if $O_t f$ or $V_t f$ converges in the L^1-norm on each compact set, then these limits are (a.e. equal to) harmonic functions in R^n. This result implies the following generalization of a recent theorem due to Delsarte and Lions [Comment. Math. Helv. **33** (1959), 59–69; MR **21** #1461]: If f is infinitely differentiable in R^3 and if two positive numbers s_1 and s_2 exist with s_1/s_2 irrational so that $O_{s_1} f$ and $O_{s_2} f$ are harmonic in R^3, then f is itself harmonic in R^3.

Next the author proves a pair of theorems which yield the following characterizations for functions harmonic in R^n. First let $0 \leqq a < b$, $n \geqq 1$. (1) If $O_t f$ is independent of t for $a<t<b$, then f is harmonic. (2) If $V_t f$ is independent of t for $a<t<b$, then f is harmonic. (3) If $O_t f=V_t f$ a.e. in $R^n \times (a, b)$, then f is harmonic. Next let $a \geqq 0$. (4) If $n \geqq 2$ and if $O_t f$ is almost periodic (or in particular periodic) in $a<t<\infty$, then f is harmonic. (5) If $n \geqq 1$ and if $V_t f$ is almost periodic in $a<t<\infty$, then f is harmonic. The author discusses how these characterizations are related to a recent theorem due to Delsarte [C. R. Acad. Sci. Paris **246** (1958), 1358–1360; MR **20** #2548]. *F. W. Gehring* (Ann Arbor, Mich.)

5829:

Harmegnies, P. Conditions pour qu'une famille de surfaces à un paramètre représente les équipotentielles du champ d'un système possible de conducteurs. Acad. Roy. Belg. Bull. Cl. Sci. (5) **46** (1960), 70–74.

Necessary and sufficient conditions are given that a family of surfaces be equipotentials. The results are illustrated in the classical case of confocal ellipsoids. *E. Pinney* (Berkeley, Calif.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also B6607, B6654.

5830a:

Boole, George. ★Calculus of finite differences. Edited by J. F. Moulton. 4th ed. Chelsea Publishing Co., New York, 1957. xii+336 pp. $1.39.

5830b:

Boole, George. ★A treatise on the calculus of finite differences. 2nd (last revised) ed. Edited by J. F. Moulton. Dover Publications, Inc., New York, 1960. xii+336 pp. $1.85.

These are both unaltered reprintings in paper covers of the 1872 edition of the same book [cf. MR **20** #1124].

5831:

Miller, Kenneth S. ★An introduction to the calculus of finite differences and difference equations. Henry Holt and Co., New York, 1960. viii+167 pp. $4.50.

This book gives a concise and clearly written introduction to certain topics in the Calculus of Differences, which include difference and sum operators; infinite products and the sine and gamma functions; Bernoulli's polynomials and the Euler-Maclaurin formula; linear difference equations.

L. M. Milne-Thomson (Madison, Wis.)

5832:

Tauber, Selmo; Dean, Donald. On difference equations containing step and delta functions. J. Soc. Indust. Appl. Math. 8 (1960), 174–180.

Using the difference operators Δ, Δ^{-1}, I, and E, relations are obtained between the Kronecker delta δ_x^a and a Heaviside-type function $\psi(x)=0$ for $x<0$ and $\psi(x)=1$ for $x\geq 0$, defined only for integer values of x, i.e., $\Delta x=1$. These operators and relations and the binomial expansion are used to obtain the complementary, particular and complete solution for the first order equations

$$(\Delta-\alpha I)f(x) = \psi(x-a) \quad \text{and} \quad (\Delta-\alpha I)f(x) = \delta_x^{a-1}.$$

Linear second and third order equations with constant coefficients and the same in homogeneous terms are also discussed. *G. W. Evans, II* (Menlo Park, Calif.)

5833:

Naftalevič, A. G. On meromorphic solutions of a difference equation. Uspehi Mat. Nauk 14 (1959), no. 4 (88), 195–202. (Russian)

This paper is concerned with the existence and order of meromorphic solutions of the difference equation

$$\sum_{k=1}^{n} a_k f(z+\alpha_k) = g(z) \quad (a_k \neq 0;\ k = 1, 2, \cdots, n),$$

where a_k and α_k are given complex numbers and $g(z)$ is a given meromorphic function. The principal results established are the following. (1) The poles and corresponding principal parts of a meromorphic solution of the equations are determined uniquely when we are given the principal parts of the solution and the corresponding poles in a fundamental strip. (2) If $g(z)$ is a meromorphic function of order ρ, the equation has a meromorphic solution whose order does not exceed $\rho+n-1$.

L. M. Milne-Thomson (Madison, Wis.)

5834:

Babuška, Ivo. The Fourier transform in the theory of difference equations and its applications. Arch. Mech. Stos. 11 (1959), 349–381. (Polish and Russian summaries)

This long and important paper is the first of a series describing work, initiated by the author, directed towards the establishment of a theory of the Fourier transforms of functions defined at the node-points of a net, and the systematic application of such a theory to the solution of boundary value problems in the theory of difference equations. It aims at providing the analogue for finite difference equations of methods which have been successfully applied in the solution of boundary value problems for partial differential equations.

In the present paper this theory is developed for the one-dimensional case, using the theory of distributions.

In § 2 several concepts used in the sequel are defined, e.g., R is the linear space consisting of all complex functions defined on the set M of (signed) integers such that to any $f\in R$, there exists a positive constant C and a positive integer p such that $f(n)\leq C(|n|^p+1)$, $n\in M$. The map A from R to H, the set of all complex-valued functions defined on M, defined by the equation $(Af)(n)=\sum_{m=-\infty}^{\infty} f(m)a(n-m)$ is called the 'convolution transform'. The A-problem in R is then taken to mean finding a $g\in R$, such that given $f\in R$, (*) $Ag=f$.

In § 3 examples are given of the occurrence of A-problems in mechanics (problems concerning an inelastic thread on elastic supports, a continuous beam on elastic supports, a girder frame) and potential theory (finite difference Poisson equation in an infinite strip).

In §§ 4, 5 the theory is given in full. Defining U to be a space consisting of all generalized functions $f(x)$ which can be expressed as a series $\sum_{n=-\infty}^{\infty} c_n e^{-inx}$, converging in the sense of generalized functions, the author defines the Fourier transform of R to be the mapping $\mathscr{F}$ of R on U such that $\mathscr{F}g=\sum_{n=-\infty}^{\infty} g(n)e^{inx}$, $g\in R$. The main results proved are (1) that, under wide conditions, $\mathscr{F}Af=\mathscr{F}a\mathscr{F}f$, and (2) that, in general, for any $f\in R$, equation (*) has precisely one solution $g=A^{-1}f$, where A^{-1} is the convolution transform defined by the function $b=\mathscr{F}^{-1}(1/\mathscr{F}a)$.

In § 6 certain of the A-problems posed in § 3 are discussed. In the case of one of them (Poisson problem in an infinite strip) the similarity of the procedure with that in the differential equation case is discussed.

I. N. Sneddon (Glasgow)

5835:

Vitásek, Emil. The n-dimensional Fourier transform in the theory of difference equations. Arch. Mech. Stos. 12 (1960), 185–202. (Polish and Russian summaries)

This is the second paper in the series initiated by Babuška [see preceding review]. In it the theory of Fourier transforms of mesh functions, developed by Babuška in the one-dimensional case, is generalized to the n-dimensional case. The results go over in an obvious way and the basic ideas of the proofs are, in most cases, similar to those of the ealier paper. The use of the theory is illustrated by a discussion of three problems in applied mathematics—an inelastic plane network on elastic supports; an infinite girder on elastic supports; Green's function for the difference equation in n variables.

I. N. Sneddon (Glasgow)

5836:

Mirolyubov, A. A. A difference equation of infinite order. Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo, pp. 207–216. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

The equation considered is

$$(*)\quad M[f(x)] \equiv (x+b)f(x-H) + \sum_{k=0}^{\infty} (a_k x+b_k)f(x-h_k) = F(x),$$

subject to the conditions: (1) $0\leq h_k\leq H$ $(k=0, 1, \cdots)$; (2) $a_0\neq 0$, $h_0=0$; (3) neither 0 nor H is a limit point of $\{h_k\}$; (4) $\sum a_k$, $\sum b_k$ are absolutely convergent. Equation

(*) with $F(x)=1/(z-x)$ is shown to have two solutions expressible in a Laplace-type of contour integral $f_j(x, z)=\int \gamma(t, z)e^{xt}dt$ $(j=1, 2)$, where the contours are over two (different) infinite rays, and where γ satisfies the differential equation

$$d\gamma/dt+\gamma[\Theta'(t)-\Theta_1(t)]/\Theta(t) = -e^{-zt}.$$

Here Θ, Θ_1 are the entire functions

$$\Theta(t) = e^{-Ht}+\sum_{k=0}^{\infty} a_k e^{-h_k t},$$

$$\Theta_1(t) = be^{-Ht}+\sum_{k=0}^{\infty} b_k e^{-h_k t}.$$

These solutions f_1, f_2 are valid over different half-planes. If $F(x)$ is analytic in a region G, a decomposition $F(x)=F_1(x)+F_2(x)$ is found, such that $M[T_j(x)]=F_j(x)$ has a solution T_j $(j=1, 2)$ that can be expressed in terms of f_j (above); and the function $T(x)=T_1+T_2$ is shown to satisfy equation (*) in G. The homogeneous equation $(F(x)\equiv 0)$ is also treated.

I. M. Sheffer (University Park, Pa.)

5837:

Straus, E. G. A functional equation proposed by R. Bellman. Proc. Amer. Math. Soc. **10** (1959), 860–862.

This paper gives an elegant solution of Research problem 14 of R. Bellman [Bull. Amer. Math. Soc. **64** (1958), 178–179] asking for (analytic) solutions of the functional equation

$$f(uv, (uv)', \cdots, (uv)^{(n)}) = f(u, u', \cdots, u^{(n)})+f(v, v', \cdots, v^{(n)}),$$

where $u(x)$, $v(x)$ are arbitrary nonzero n-times differentiable functions. The author proves, without imposing any analyticity condition on f, that $f(u, u', \cdots, u^{(n)})=\sum_{k=0}^{n} a_k(d^k \log u/dx^k)$ {log $|u|$ would be more correct}, where a_k $(k=0, 1, \cdots, n)$ are arbitrary additive functions (so under supposition of, e.g., boundedness, $a_k(t)=a_k t$). The author mentions several generalizations: f depends also on x, uv is replaced by $g(h(u)+h(v))$ (it turns out that $g(u)=h^{-1}(u+c)$), differentiation is replaced by other linear operators, f may depend on an infinite number of variables, etc.

The paper of the author, M. Hoczu and reviewer forthcoming in Publ. Math. Debrecen should be mentioned here in connection with the above problem and the problems 15–17 of R. Bellman [ibid.].

J. Aczél (Debrecen)

5838:

Rufener, E. Quasiarithmetische Mittelbildungen an Verbindungsrenten. Mitt. Verein. Schweiz. Versich.-Math. **59** (1959), 241–250. (French, Italian and English summaries)

The author applies the theory of quasiarithmetic means [see, e.g., G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd ed., Cambridge Univ. Press, 1952; MR **13**, 727; Chapter III; author quotes later works of H. Jecklin, Elem. Math. **3** (1948), 12–17; Comment. Math. Helv. **22** (1949), 260–270; Elem. Math. **4** (1949), 111–115; MR **9**, 502; **10**, 685; **11**, 235] to survival functions. He does not state his results explicitly, but the main result seems to be the following: If the function $f(x)$ is logarithmically concave (in the narrow sense) and has a continuous derivative, and further

$$(1)\quad \int_0^{\infty} \prod_{j=1}^{n} (f(x_j+t)/f(x_j))\, dt = h(m(x_1, x_2, \cdots, x_n)),$$

where $m(x_1, x_2, \cdots, x_n)$ is derivable and satisfies the functional equation

$$(2)\quad m(x_1+t, x_2+t, \cdots, x_n+t) = m(x_1, x_2, \cdots, x_n)+t,$$

then either $f(x)=ka^xb^{x^2}$ (survival function of Gauss) or $f(x)=ka^xb^{c^x}$ (survival function of Makeham). The essential step of the proof is that for $g(x)=f'(x)/f(x)$

$$(3)\quad m(x_1, x_2, \cdots, x_n) = g^{-1}\Big(\sum_{j=1}^{n} g(x_j)/n\Big)$$

must hold. Then the differential equation $g''(x)=Cg'(x)$ is deduced for g. {So this proof makes use of third order derivability of the function f. Nevertheless it is well known [cf. Hardy, Littlewood, and Pólya, loc. cit., § 3.3; or M. Nagumo, Japan. J. Math. **7** (1930), 71–79] that (2) and (3) can hold for continuous strictly monotonic functions if and only if $g(x)=A+Bx$ or $g(x)=A+Bc^x$ and so the theorem holds in the above stated form too. Also some further conditions of the author prove to be unnecessary.}

Generalizations are made for the cases where in (1) n functions f_j $(j=1, 2, \cdots, n)$ appear instead of f {here a similar shortening of the proof and reduction of conditions can be achieved by the method of reviewer's paper in Publ. Math. Debrecen **7** (1960), 10–15 [MR **22** #7077]}, and for

$$h(m_1(x_1, \cdots, x_n), m_2(x_1, \cdots, x_n), \cdots, m_k(x_1, \cdots, x_n))$$

on the right-hand side of (1) (then in (2) m_i $(i=1, 2, \cdots, k)$ appear instead of m). In the former case the result is the same, the latter leads to the differential equation of A. Quiquet [Bull. Inst. Actuaires Français **3** (1893), 97–186] for g.

{Some misprints: in (5) and (24) ∂x_r should appear instead of $\partial \chi_r$, in (20′) k_λ instead of k.}

J. Aczél (Debrecen)

SEQUENCES, SERIES, SUMMABILITY

See also 5758.

5839:

Ford, Walter B. ★Studies on divergent series and summability & The asymptotic developments of functions defined by Maclaurin series. Chelsea Publishing Co., New York, 1960. x+342 pp. $6.00.

Reproduction, with a handful of corrections supplied by the author, of two books originally published in 1916 and 1936 respectively.

5840:

Mařík, Jan; Neubauer, Miloš. Reihen mit nichtnegativen Gliedern. Časopis Pěst. Mat. **85** (1960), 188–197. (Czech. Russian and German summaries)

Let $\mathfrak{P}$ be the set of all sequences $\{a_n\}_{n=1}^{\infty}$ of nonnegative real numbers; $\mathfrak{B}$ all bounded sequences in $\mathfrak{P}$; $\mathfrak{K}$ all monotone decreasing sequences in $\mathfrak{P}$ with limit 0; $\mathfrak{K}_\infty$ all $\{b_n\}\in\mathfrak{K}$ such that $\sum b_n=\infty$. Let $\mathfrak{M}$ be the set of all

$\{a_n\} \in \mathfrak{P}$ such that $\{\min(a_n, b_n)\} \in \mathfrak{R}_\infty$ if $\{b_n\} \in \mathfrak{R}_\infty$. Theorem 1: A sequence $\{a_n\}$ in $\mathfrak{P}$ belongs to $\mathfrak{M}$ if and only if there is an increasing sequence $\{k_n\}$ of positive integers such that $\{k_n/n\} \in \mathfrak{B}$, $a_{k_n} > 0$, and $\{1/k_n a_{k_n}\} \in \mathfrak{B}$. Theorem 2: A sequence $\{a_n\}$ in $\mathfrak{R}$ belongs to $\mathfrak{M}$ if and only if $a_n > 0$ and $\{1/na_n\} \in \mathfrak{B}$. Theorem 3: If $\{a_n\} \in \mathfrak{R} \cap \mathfrak{M}$ and $\sum a_{n_k} < \infty$, then the upper density of the set $\{n_1, n_2, \cdots\}$ is zero. Other related results are also obtained.

E. Hewitt (Seattle, Wash.)

5841:

Stipanić, Ernest. Due teoremi su alcune serie divergenti di Dini a termini positivi. Boll. Un. Mat. Ital. (3) **14** (1959), 516–524.

If $\sum_{n=0}^\infty C_n = S$ $(C_{n-1} > C_n > 0)$ is convergent and $\sum_0^\infty d_n(d_{n-1} > d_n \to 0,\ n \to \infty)$ is divergent, then Dini has shown that (1) $\sum_1^\infty (C_{n-1} - C_n)/C_{n-1}$; $\sum_0^\infty C_n/r_{n-1}$; (2) $\sum_1^\infty (d_{n-1} - d_n)/d_{n-1}$; $\sum_0^\infty d_n/D_n$ $(r_{n-1} = \sum_{R=n}^\infty C_R$, $D_n = \sum_{R=0}^n d_R)$ are divergent series. In the present paper the author proves some theorems concerning series (1) and (2). The following theorem is typical. If $\mathrm{Lim}_{n\to\infty} (C_n/C_{n-1}) = q < 1$ $((C_n/C_{n-1}) < 1,\ n \geqq n_0)$ and if $C_n^{v+1} \leqq C_{n+v}(C_{n-1})^v$ $(n \geqq 1,\ v \geqq 0)$, then the series

$$\sum_{n=1}^\infty \left(\frac{C_{n-1} - C_n}{C_{n-1}} - \frac{C_n}{r_{n-1}}\right)$$

is convergent and its sum H satisfies

$$H = \sum_{n=1}^{n_0-1} \left(\frac{r_n}{r_{n-1}} - \frac{C_n}{C_{n-1}}\right) + \theta \frac{C_{n_0-1}}{r_{n_0-1}} \left(\frac{q}{1-q} - \frac{r_{n_0-1}}{C_{n_0-1}}\right)$$

for some $0 \leqq \theta < 1$.

V. F. Cowling (Lexington, Ky.)

5842:

Šalát, Tibor. On an application of continued fractions in the theory of infinite series. Časopis Pěst. Mat. **84** (1959), 317–326. (Czech. Russian and English summaries)

Let $\sum_{k=1}^\infty u_k$ be a series of real numbers, and $M_k = (c_1^k, c_2^k, \cdots, c_n^k, \cdots)$, $k = 1, 2, 3, \cdots$, a sequence of real numbers. Let $A_k = \sup |c_n^k|$, $n = 1, 2, 3, \cdots$, and

$$\sum_{k=1}^\infty A_k |u_k| < \infty.$$

Let

$$\text{(A)} \qquad x = \frac{1}{n_1 +} \frac{1}{n_2 +} \frac{1}{n_3 +} \cdots$$

be the infinite expansion of the irrational number $x \in (0, 1)$ into a continued fraction. Let the function ϕ be defined as follows: $\phi(0) = 0$, $\phi(x) = \sum_{i=1}^\infty c_{n_i}{}^i u_i$, if x is irrational with the expansion (A). If x is rational, $0 < x \leqq 1$, then its expansion into a continued fraction is finite, of the type (A). In the latter case, $\phi(x) = \sum_{i=1}^r c_{m_i}{}^i u_i$. By this definition the set of all values of the function ϕ on the set of all irrational numbers of the interval (0, 1) is identical with the set of all sums of the series $\sum_1^\infty \varepsilon_k u_k$, where ε_k is a member of the sequence M_k. Then the author gives the following theorems: (1) The function ϕ is integrable on (0, 1) in the Riemann sense. (2) Let $\sum_{k=1}^\infty u_k$ be a series of real numbers, let the sequence $\{M_k\}$ be normal with respect to this series. For every $x \in (0, 1)$ except the points of a certain set, $\lim \sup_{k\to\infty} \phi_k(x) = +\infty$, $\lim \inf_{k\to\infty} \phi_k(x) = -\infty$.

E. Frank (Chicago, Ill.)

5843:

Prasad, B. N.; Pati, T. The second theorem of consistency in the theory of absolute Riesz summability. Math. Ann. **140** (1960), 187–197.

The authors improve their "second theorem of consistency" for absolute Riesz summability of non-integral order κ [Trans. Amer. Math. Soc. **85** (1957), 122–133; MR **19**, 135], which states that $|(R, \lambda_n, \kappa)| \subset |(R, \varphi(\lambda_n), \kappa)|$. Their former conditions on the function $\varphi(u)$ are replaced by weaker ones.

G. G. Lorentz (Syracuse, N.Y.)

5844:

Berg, Lothar. Abelsche und Taubersche Sätze für Potenzreihen mit Fehlerabschätzungen. Math. Nachr. **20** (1959), 25–38.

In a previous work the author has given a general form of the Laplace method for the asymptotic evaluation of integrals [Math. Nachr. **17** (1958), 101–135; MR **20** #5390]. He now adds to the asymptotic formulas a remainder estimate. The conditions are quite complicated, but the following special case gives the flavor. Let χ be a function of class C^4 as $x \to \infty$ such that $\chi^{(k)}(x) \sim c_k x^{\varepsilon-k}$, $1 \leqq k \leqq 4$, with $0 < \varepsilon < 1$ and $c_2 > 0$. Then as $s \uparrow 1$

$$\sum_{n=0}^\infty e^{-\chi(n)} s^n = \left\{\frac{2\pi}{\chi''(x)}\right\}^{1/2} e^{-\chi(x)} s^x \{1 + r(x)\},$$

where $r(x) = O(x^{-\varepsilon} \log x)$ or $r(x) = O(x^{-1+\varepsilon/2})$ according as $0 < \varepsilon \leqq 2/3$ or $2/3 < \varepsilon < 1$, and $\log s = \chi'(x)$. The author also obtains simple Tauberian theorems of the following character. If $\sum a_n s^n \sim \sum b_n s^n$ as $s \uparrow 1$ and $a_n \geqq b_n$, then $\sum_0^n a_k \sim \sum_0^n b_k$. Numerous examples are worked out in detail.

J. Korevaar (Madison, Wis.)

5845:

Russell, Dennis C. Summability of power series on continuous arcs outside the circle of convergence. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 1006–1030. (French summary)

In the earlier work on summability of power series, matrices were constructed (usually of Mittag-Leffler type) which summed the series to its generating function in a simply-connected domain extending outside the circle of convergence [see the reviewer's *Infinite matrices and sequence spaces*, Macmillan, London, 1950; MR **12**, 694; pp. 181–200]. The reviewer and P. Dienes [Proc. London Math. Soc. (2) **45** (1938), 45–63] first gave examples of matrices which sum $\sum z^n$, and a restricted class of power series, at isolated points (as opposed to domains) outside the circle of convergence. Further examples of this were given by P. Vermes [Proc. Edinburgh Math. Soc. (2) **8** (1947), 1–13; Acta Sci. Math. Szeged **14** (1951), 23–38; Colloque sur la théorie des suites, Brussels, (1957), pp. 60–86, Gauthier-Villars, Paris, 1958; same Bull. (5) **44** (1958), 830–838; MR **9**, 234; **13**, 27; **20** #7163, 7164] and S. E. Tolba [Nederl. Akad. Wetensch. Proc. Ser. A **55** (1952), 380–387; MR **14**, 36]. In the present paper the author, on a suggestion by the reviewer, investigates what is, in a sense, an intermediate stage between domains and isolated points, namely, summability of power series on continuous arcs outside the circle of convergence. Examples are given of matrices which sum $\sum z^n$ on isolated lines or sections of lines outside the circle of convergence, and also two general theorems which enable extension to

be made to a class of Taylor series whose generating function has a single singularity (either a pole or a branch-point) of a certain type; this class of series includes some binomial series. A procedure of Vermes can also be used to give an extension to a class of meromorphic functions. The proofs given of these results are of an interesting character. *R. G. Cooke* (London)

APPROXIMATIONS AND EXPANSIONS

5846:

Capra, Vincenzo. Sul resto delle formule d'interpolazione espresso mediante operatori incrementali. Univ. e Politec. Torino. Rend. Sem. Mat. **18** (1958/59), 137–158.

Formulas for the remainder in polynomial interpolation of a continuous function are expressed in terms of differences. For example, if f is a function of one variable, and P_n is of degree n, with $P_n(x_i)=f(x_i)$ at distinct points $x_0<x_1\cdots<x_{n-1}<x_n$, then on the interval (x_0, x_n),

$$f(x)-P_n(x) = \prod (x-x_i)\mathcal{D}_\omega{}^{n+1}f(t)/(n+1)!$$

for sufficiently small ω, and suitable t on (x_0, x_n), where $\mathcal{D}_\omega f(x)=[f(x+\omega)-f(x-\omega)]/2\omega$. Modifications are indicated for the case of coincident abscissas and for the case of functions of two variables. There is a concluding section on numerical evaluation of bounds on the remainder, with two examples. *L. M. Graves* (Chicago, Ill.)

5847:

Pulatov, A. I. On some classes of orthonormal systems. Izv. Akad. Nauk UzSSR. Ser. Fiz.-Mat. **1958**, no. 6, 57–64. (Russian. Uzbek summary)

The author studies two orthonormal systems in the interval $(0, \pi)$ obtained by orthogonalizing in this interval the sets [cos nx] and [sin nx] with a given weight function. The substitution $t=\cos x$ transforms the interval $(0, \pi)$ into $(-1, 1)$ and reduces the question to well-known orthonormal systems of algebraic polynomials with a weight function. Recurrence relations and Christoffel's formulas are deduced and some theorems on convergence of Fourier series in these systems are proved.

E. Kogbetliantz (New York)

5848:

Russo, Salvatore. Un criterio di convergenza delle serie di polinomi di Legendre. Boll. Un. Mat. Ital. (3) **15** (1960), 20–24. (English summary)

The convergence of the Legendre series $\sum_{n=0}^{\infty} a_n P_n(x)$ at an interior point x_0, $-1<x_0<1$, of the basic interval is investigated. The assumption is made that the partial sums of the corresponding power series $\sum_{n=0}^{\infty} a_n t^n$ are bounded on an appropriate arc of a certain circle.

G. Szegö (Stanford, Calif.)

5849:

Riekstyn'š, É. [Riekstiņš, E.]. Asymptotic expansions for the real roots of certain transcendental equations. Latvijas Valsts Univ. Zinātn. Raksti **28** (1959), no. 4, 67–86. (Russian. Latvian summary)

The real roots of equations of the type

$$F(x) \equiv \sin(x+\omega)f_1(x)+\cos(x+\omega)f_2(x)-f_3(x) = 0$$

are described for large x when the f_i have asymptotic series of the form $\sum a_k x^{\alpha_k}$ as $x\to\infty$ ($\alpha_k<\alpha_{k+1}$, $\alpha_k\to\infty$ as $k\to\infty$). If f_i' is continuous and has the same type of representation as f_i, then $F(x)=\Phi(x)+o(x)$ as $x\to\infty$, and the real roots of Φ are asymptotic to roots of F. The coefficients of the asymptotic series for the real roots of F are determined. Application is made to Sturm-Liouville problems and to functions F which are a sum of products of Bessel functions. *N. D. Kazarinoff* (Moscow)

FOURIER ANALYSIS

5850:

Adamović, Dušan. Généralisations de deux théorèmes de Zygmund-B. Sz.-Nagy. Acad. Serbe Sci. Publ. Inst. Math. **12** (1958), 81–100.

Let the functions $f(x)$, $g(x)$ be non-increasing and bounded below in $(0, \pi)$, $f(x)\in L(0, \pi)$, $xg(x)\in L(0, \pi)$. Let $\{a_n\}$ be the coefficients in the cosine series of $f(x)$, $\{b_n\}$, in the sine (not necessarily Fourier) series of $g(x)$. The author establishes a number of necessary and sufficient conditions connecting integrability with the absolute convergence of series involving the coefficients. Thus, for $0<\gamma<2$, the absolute convergence of $\sum n^{-\gamma}L(n)b_n$ is equivalent to $x^{\gamma-1}L(1/x)g(x)\in L(0, \pi)$, where $L(x)$ is any function of "slow growth" in the sense of Karamata. Mutatis mutandis, this holds for $f(x)$ provided $0<\gamma<1$. For $\gamma=1$ a similar result holds if the functions $f(x)$ and $L(x)$ satisfy additional conditions involving logarithmic factors. Similar conclusions are drawn for the cases in which $x^{-\alpha}L(x)$ is non-decreasing for various ranges of α and for the special values $\alpha=1$, $\alpha=2$.

L. Lorch (Edmonton, Alta.)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

5851:

Goldberg, Richard R. Convolutions and general transforms on L^p. Duke Math. J. **27** (1960), 251–259.

In a previous paper [Proc. Amer. Math. Soc. **10** (1959), 385–390; MR **21** #4329] the author proved that if f and g are a pair of cosine transforms in $L_2(0, \infty)$, then so are

$$F(x) = \int_0^\infty \frac{1}{y}\,\varphi\left(\frac{x}{y}\right)f(y)\,dy,\quad G(y) = \frac{1}{y}\int_0^\infty \varphi\left(\frac{x}{y}\right)g(x)\,dx,$$

where φ is a non-negative function with the property $\int_0^\infty \varphi(y)y^{-1/2}dy<\infty$. It was E. C. Titchmarsh who originally gave the result for $\varphi(x)=1$ in $0\leq x\leq 1$ and $\varphi(x)=0$ in $1<x<\infty$ [*Introduction to the theory of Fourier integrals*, Clarendon, Oxford, 1937, p. 93]. In the present paper the author shows that his previous generalization applies not only to the cosine transform but to a class of general transforms including Watson transforms in L^p, $1<p\leq 2$ [I. Busbridge, Quart. J. Math. Oxford Ser. **9** (1938), 148–160; H. Kober, ibid. **9** (1938), 41–52; S. Bochner and K. Chandrasekharan, *Fourier transforms*, Princeton Univ. Press, Princeton, N.J., 1949; MR **11**, 173; Chap. V]. *K. Chandrasekharan* (Bombay)

5852:

Дёч, Г. [Doetsch, Gustav]. ★Руководство к практическому применению преобразования Лапласа. [Anleitung zum praktischen Gebrauch der Laplace-Transformation]. Translated from the German by G. A. Vol′pert. 2nd ed. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. 207 pp. 5.60 r.

The original was reviewed in MR **19**, 139.

5853:

Avetisyan, A. E. Two theorems on functions analytic in angular domains. Akad. Nauk Armyan. SSR. Dokl. **29** (1959), 193–202. (Russian. Armenian summary)

The author considers a generalized Laplace transform for functions that are regular and of mean type of order ρ in an angle, and its inversion by an integral involving Mittag-Leffler functions. He shows, first, that some of the theory of the Pólya indicator can be extended to this case; second, that under certain circumstances the inversion integral converges in a larger region than the original angle. [Cf. A. J. Macintyre, Proc. London Math. Soc. (2) **45** (1938), 1–20.]

R. P. Boas, Jr. (Evanston, Ill.)

5854:

Freudenthal, Hans. Operators—from Heaviside to Mikusiński. Simon Stevin **33** (1959), 3–19. (Dutch)

This is a brief sketch of the history of operational calculus. The use of symbolic methods is traced from Leibniz, through Lagrange, Brisson, Servois, Fourier and Cauchy (!) in France, and Gregory, de Morgan, Murphy and Boole in England, to Heaviside. It is explained that the self-taught Heaviside was greatly influenced by the work of Fourier and Boole, whereas there was little mathematical communication between him and his contemporaries; the time certainly was not ripe for abstract analysis and algebra. Thus Heaviside is presented as a lonely "operator", who had the courage to apply symbolic calculus to all his problems. A few examples are indicated. The author then deals briefly with the Laplace transformation as developed by Bromwich, Carson, Doetsch and others. His opinion: advocating the Laplace transformation instead of Heaviside's method is like suggesting that one study Sanskrit in place of Basic English. This brings him to what he considers the recent great and definitive accomplishments in the field, those of Mikusiński. [See the latter's *Rachunek operatorów*, Polskie Towarzystwo Matematyczne, Warsaw, 1953; English edition, Pergamon, 1959; MR **16**, 243; **21** #4333.] He gives a brief exposition and ends with a {perhaps premature} call to all mathematicians to drive out the Laplace transformation!

J. Korevaar (Madison, Wis.)

5855:

Bose, S. K.; Mehra, A. N. On Meijer transform of two variables. Gaṇita **9** (1958), 43–64.

In 1941 Meijer [Nederl. Akad. Wetensch. Proc. **44** (1941), 727–737, 831–839; MR **3**, 38, 109] devised a transform with a Whittaker function in the kernel. The present authors generalize this transform to two variables. Term by term transformation of the series is considered and conditions for its validity are given. Several integral properties of the transforms are derived.

D. J. Hofsommer (Amsterdam)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

5856:

Bădescu, Radu. Sur une équation intégrale linéaire. Rev. Math. Pures Appl. **4** (1959), 221–231.

The paper studies the integral equation

$$\Phi(z)-\mu\int_C\frac{\Phi(z)-\Phi(s)}{|z-s|}\,ds=f(z).$$

Polynomial solutions are obtained in determinant form. The series of polynomials belongs to a real or complex Banach space (E), for any value of μ interior to a certain compact connection $\bar{\Delta}$. The integral equation is a special case of a Fredholm equation of the third type.

D. E. Spencer (Storrs, Conn.)

5857:

Picone, Mauro. Sur les équations intégrales linéaires de deuxième espèce de Volterra avec noyau de translation. C. R. Acad. Sci. Paris **250** (1960), 46–48.

This is a short note containing several results concerning certain translation type Volterra integral equations for matrix valued functions.

C. R. DePrima (Pasadena, Calif.)

5858:

Picone, Mauro. Nuove determinazioni concernenti l'equazione integrale non lineare di Volterra. Ann. Mat. Pura Appl. (4) **50** (1960), 97–113. (English summary)

The non-linear integral equation in question is $u(x)=f(x)+\int_{x_0}^x K[x,y,u(y)]dy$, where u, f and K are vector functions. It is assumed that K is of the form $H(x,y,u)-p(x)G(y,u)[u-g(y)]$, where p and G are $n\times n$ diagonal matrix functions, pG has non-negative entries, and p, G and H satisfy suitable boundedness restrictions. The following modified method of successive approximations is used. One defines $u^{(n+1)}$ as the solution of the linear Volterra equation

$$u^{(n+1)}(x)=f(x)+\int_{x_0}^x H[x,y,u^{(n)}(y)]dy-\int_{x_0}^x p(x)G(y,u^{(n)}(y))[u^{(n+1)}(y)-g(y)]dy.$$

One then shows that the $u^{(n)}$ are uniformly convergent in a suitable domain of n-space to a solution of the given equation. The variables may be real or complex.

G.-C. Rota (Cambridge, Mass.)

5859:

Pohožaev, S. I. Analogue of Schmidt's method for non-linear equations. Dokl. Akad. Nauk SSSR **136** (1961), 546–548 (Russian); translated as Soviet Math. Dokl. **2**, 103–105.

The author gives a method for the reduction of the solution of non-linear integral equations to the solution of integral equations with degenerate kernels.

H. P. Thielman (Oxnard, Calif.)

5860:

Riekstyn′š, È. [Riekstiņš, E.]; Zolberga, R. Asymptotic expansions for the eigenvalues and eigenfunctions of a Sturm-Liouville problem for an integro-differential equation for large values of the parameter. Latvijas Valsts

Univ. Zinātn. Raksti **28** (1959), no. 4, 87–94. (Russian. Latvian summary)

The asymptotic behavior of the eigenvalues and eigenfunctions of the problem

$$u''+[\lambda^2-\omega(t)]u = \int_0^t K(t, \tau)u(\tau, \lambda)d\tau,$$

$$u(0, \lambda) = u(l, \lambda) = 0$$

is determined for λ large. It is assumed that ω and K are infinitely differentiable for t and τ on $[0, l]$.

N. D. Kazarinoff (Moscow)

5861:

Halanay, Aristide. Solutions périodiques des systèmes généraux à retardement. C. R. Acad. Sci. Paris **250** (1960), 3557–3559.

Existence, uniqueness, and boundedness theorems are given for solutions to the integro-differential equation

$$\dot{x}(t) = \int_{-\infty}^0 x(t+s)d_s\eta(t, s)+f(t)$$

for x and f vectors and η a square matrix.

E. Pinney (Berkeley, Calif.)

FUNCTIONAL ANALYSIS

See also 5695, 5715, B6062.

5862:

Konda, Tomoko. On quasi-normed space. I, II. Proc. Japan Acad. **35** (1959), 340–342, 584–587.

A quasi-norm with power r in a linear space L is like a norm except that homogeneity of order 1 is replaced by: For each real λ and each x in L, $\|\lambda x\| = |\lambda|^r\|x\|$. Such a space is, of course, a locally bounded linear metric space. Note I discusses completion of such a space. Note II considers finite dimensionality and compactness, and also the notion of norm of a linear operator between such spaces.

M. M. Day (Urbana, Ill.)

5863a:

Isêki, Kiyoshi. An approximation problem in quasi-normed spaces. Proc. Japan Acad. **35** (1959), 465–466.

5863b:

Isêki, Kiyoshi. On finite dimensional quasi-norm spaces. Proc. Japan Acad. **35** (1959), 536–537.

In the same setting as above [see review #5862] the first note proves that the distance from a point of L to a finite dimensional subspace E is attained at some point of E. The second note reproves this small part of the standard theorem that all n-dimensional linear topological spaces are isomorphic.

M. M. Day (Urbana, Ill.)

5864:

Pták, Vlastimil. On the closed graph theorem. Czechoslovak Math. J. **9 (84)** (1959), 523–527. (Russian summary)

The author continues his analysis of the open mapping theorem [same J. **3 (78)** (1953), 301–364; Bull. Soc. Math. France **86** (1958), 41–74; MR **16**, 262; **21** #4345]. A mapping f of a space E onto a space F is said to be nearly open provided for each point p of E and each neighborhood U of p, the closure of fU is a neighborhood of fp. A convex (=locally convex Hausdorff linear) space E is said to be B-complete provided each continuous nearly open linear mapping f of E onto a convex space is in fact open. If this is assumed only when f is one-to-one, then E is said to be B_r-complete. Using the formalism of the duality theory, the author establishes the following result: Suppose E and F are convex spaces and G a closed linear subspace of $E\times F$ such that $P_FG=F$ (where $P_F(x, y)=y$ for $(x, y)\in E\times F$). Assume that E is B-complete and that the following condition is fulfilled: for each neighborhood U of 0 in E, the closure of the set $G_FU=P_F((U\times F)\cap G)$ is a neighborhood of 0 in F. Then for each such U, the set G_FU is itself a neighborhood of 0 in F.

An immediate consequence of the above result is that if E and F are convex spaces, E is B-complete, E_0 is a linear subspace of E, f is a linear mapping of E_0 onto F, and the graph of f is closed in $E\times F$, then f is open if and only if f is nearly open. There are analogous results for spaces which are B_r-complete.

Victor Klee (Copenhagen)

5865:

Cristescu, Romulus. Integrali vettoriali di Stieltjes ed operatori lineari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **27** (1959), 31–34.

This paper deals with a generalization of the classical Riesz theorem on the representation of continuous linear functionals on a space of continuous functions. In this case vector lattices are involved. Let X be a normed vector lattice; let Y be a complete vector lattice. Let T be a compact real interval. Let $C(T, X)$ be the normed space of continuous linear functions on T to X. If V is a normed linear space, let $\mathscr{L}[V, Y]$ be the class of linear mappings of V into Y such that the image of the unit ball in Y is order-bounded in Y. Then the standard representation of an element $U\in\mathscr{L}[C(T, X), Y]$ is

$$U(f) = \int_T dg(t)(f(t)),$$

where $f\in C(T, X)$ and $g: T\to\mathscr{L}[X, Y]$, with g of bounded variation in a natural sense. The proof is a straightforward adaptation of the classical proof of the Riesz theorem. There are connections with work of L. V. Kantorovič [Mat. Sb. (N.S.) **7 (49)** (1940), 209–284; MR **2**, 317].

A. E. Taylor (Los Angeles, Calif.)

5866:

Barbuti, Ugo. Sulla teoria della migliore approssimazione nel senso di Tchebychev. I. Rend. Sem. Mat. Univ. Padova **30** (1960), 82–96.

Let $\mathscr{C}$ be the Banach space of all real continuous functions on the compact Hausdorff space S (with supremum norm), and let $\mathscr{P}$ be a subspace of $\mathscr{C}$. If $f\in\mathscr{C}$ and if τ in $\mathscr{P}$ is such that $\mu=\|\tau-f\|=\inf\{\|p-f\|: p\in\mathscr{P}\}$, then (letting $E=\{x \text{ in } S: |\tau(x)-f(x)|=\mu\}$) we have (theorem 2) $\mu\leq\sup\{|f(x)|: x\in E\}$. This result is used to prove known theorems concerning interpolation in $\mathscr{P}$, uniqueness of τ (for finite dimensional $\mathscr{P}$) and related results. [Theorem 1 asserts the existence of τ for more

general subsets of $\mathscr{P}$ of $\mathscr{C}$ than finite dimensional subspaces, but the assumptions made on $\mathscr{P}$ are not clear to the reviewer.} *R. R. Phelps* (Berkeley, Calif.)

5867:

Weston, J. D. A note on the extension of linear functionals. Amer. Math. Monthly **67** (1960), 444–445.

In the paper by H. Nakano, Proc. Japan Acad. **33** (1957), 603–604 [MR **20** #2597] an extension theorem about linear functionals was stated and Mazur's theorem was proved by means of this extension theorem. The author proves conversely this extension theorem by means of Mazur's theorem, but only for the finite case. This extension theorem was correct in the paper by H. Nakano, ibid. **35** (1959), 127 [MR **21** #3743] only for the infinite case. The reviewer wants to add that the direct proof of this extension theorem is rather simple [H. Nakano, *Topology and linear topological spaces*, Maruzen, Tokyo, 1951; MR **13**, 753; § 79, proof of Theorem 2], and the corrected one is the most general form for which this proof is available.
H. Nakano (Kingston, Ont.)

5868:

Edwards, R. E. Integral bases in inductive limit spaces. Pacific J. Math. **10** (1960), 797–812.

Given a basis in a locally convex linear space E, the question arises as to whether the coefficients (which are linear functionals) are continuous. Banach established continuity in Banach spaces and Newns has done so in Fréchet spaces [Philos. Trans. Roy. Soc. London. Ser. A **245** (1953), 429–468; MR **14**, 968]. The author generalizes these results in two ways: He replaces Fréchet spaces by certain inductive limits of these; he replaces series bases by integral bases. The outstanding example of the latter is in Fourier transform theory. The generalization given below is best understood by referring closely to this case.

F is a locally convex space, E is a subspace of F endowed with its own topology, finer than the induced one. Furthermore, E is an inductive limit of Fréchet spaces. T is a locally compact space, and the basis elements are the values of a function $u: t \to u(t)$ mapping T into F. M is a locally convex vector space of Radon measures on T. The integral representation looked forward to has this form: Given $x \in E$, it determines a $\mu_x \in M$ such that $x = \int_T u(t)d\mu_x$. The precise definition of the integration procedure will be briefly suggested here. F and E are considered as embedded in F'^* (the algebraic dual of the topological dual of F). If K is a μ-measurable subset of T and y maps K into F, the WF (weak-F) integral $\int_K y(t)d\mu(t)$ exists whenever $t \to \langle y(t), y' \rangle$ is μ-integrable over K for each y' in F', and the value of the integral is then an element $z \in F'^*$. Conditions are then given (not here) to ensure that $z \in F$. The fundamental hypotheses for the paper are as follows: Given the mapping $u: t \to u(t)$ of T into F and an increasing sequence $\{K_n\}$ of compact sets in T. (I) $\int_{K_n} u(t)d\mu(t) \in F$ for each μ in M and each n. (II) There exists a subset S of F' which separates points of F and which is such that for each n and each y' in S, the map $\mu \to \int_{K_n} \langle u(t), y' \rangle d\mu(t)$ is continuous on M. (III) For each $x \in E$ there is a unique $\mu_x \in M$ such that $S_n(x) = \int_{K_n} u(t)d\mu_x(t) \in E$ for each n, and $x = \lim_{n\to\infty} S_n(x)$ in the topology of E. We state now the extension of the Banach-Newns theorem: If E is the strict inductive limit of Fréchet spaces E_a, if (I), (II), (III) hold, and if M is a Fréchet space, then (i) the mapping $X: x \to \mu_x$ is continuous, (ii) the mappings S_n $(n = 1, 2, \cdots)$ are equicontinuous from E into itself. A second theorem considers results obtained by a slight variation of hypotheses. A section is devoted to showing how series bases fit into the theory here developed of integral bases. For series, T is the set of natural numbers, $K_n = \{1, \cdots, n\}$, u is a sequence $\{u_n\}$ of elements in F (really in E) and M is (usually) the set of all scalar sequences; integrals are replaced by sums, in particular, integrals over K_n by finite sums. The author also devotes a section to dual bases. Finally, similar bases are considered. The foundation of the "similar basis theorem" is the following result: Under conditions on E which will not here be specified, the mapping $x \to \mu_x$ is an isomorphism of E onto M. Thus if the Fréchet spaces E_1 and E_2 have bases u_1 and u_2 with the same choice of M, then E_1 is isomorphic to E_2. This is the analogue of the result of Arsove [Math. Ann. **135** (1958), 283–293; MR **20** #7216]. *E. R. Lorch* (New York)

5869:

Singer, Yvan. Sur les espaces de Banach à base absolue, canoniquement équivalents à un dual d'espace de Banach. C. R. Acad. Sci. Paris **251** (1960), 620–621.

The author considers a Banach space E which has an unconditional basis ("base absolue"). His main result is the equivalence of a number of statements about E, including the following: (c) E is equivalent to the dual of a Banach space; (d) E is isomorphic to the dual of a Banach space; (f) E contains no subspace isomorphic to (c_0). He also discusses the contribution of this result towards the solution of a number of outstanding problems.

{N.B. The second summation in (e) should be to ∞.}
A. F. Ruston (Sheffield)

5870:

Gapoškin, V. F. On a property of unconditional bases in L^p space. Uspehi Mat. Nauk **14** (1959), no. 4 (88), 143–148. (Russian)

Let E be a Banach space. A sequence of elements $\{f_k\}$ each of norm 1 is an unconditional basis (Riesz basis) if, for every permutation of the f_k, every element of E has a unique expansion in a series of the f_k, converging in the E-metric. Two bases $\{f_k\}$ and $\{g_k\}$ are called equivalent if there is an operator A with continuous inverse that carries one into the other by $Af_k = g_k$. Bari [Moskov. Gos. Univ. Uč. Zap. **148** Mat. **4** (1951), 69–107; MR **14**, 289] showed that in L^2 all Riesz bases are equivalent. {A more general result can be obtained by combining results of Lorch [Bull. Amer. Math. Soc. **45** (1939), 564–569; MR **1**, 58] and Arsove [Math. Ann. **135** (1958), 283–293; MR **20** #7216].} The author shows that in $L^p(a, b)$, $1 < p \neq 2$, a and b finite, there are inequivalent Riesz bases, indeed that there is a permutation of the Haar system which is not equivalent to the Haar system. The author considered [Uspehi Mat. Nauk **13** (1958), no. 4 (82), 179–184; MR **20** #6652] the space of the coefficients of the expansions of functions of L^p in terms of a Riesz basis. He shows that these spaces for different bases coincide if and only if the bases are equivalent, and proves a more general result about total biorthogonal systems in any separable Banach space. {Still more general results than these last have been

given, in different terminology, by Arsove [paper cited above] and by Arsove and Edwards [see following review].}

R. P. Boas, Jr. (Evanston, Ill.)

5871:

Arsove, M. G.; Edwards, R. E. Generalized bases in topological linear spaces. Studia Math. **19** (1960), 95–113.

Throughout, U is a linear topological space. A basis in U is a sequence $\{x_n\}$ in U such that each x in U has a unique representation $x=\sum_{n=1}^{\infty} a_n x_n$ where the a_n are scalars. If the mapping $x \to a_n$ is continuous for each n, the basis is called a Schauder basis. A generalization of this notion due to Markuševič [Dokl. Akad Nauk SSSR **41** (1943), 227–229; Mat. Sb. (N.S.) **17** (**59**) (1945), 211–252; MR **6**, 69; **7**, 425] is the following: A (herein called) 'Markuševič basis' is a sequence $\{x_n\}$ total in U such that there exists a sequence of continuous linear functionals $\{\varphi_n\}$ biorthogonal to $\{x_n\}$ for which $\varphi_n(x)=0$ for all n implies $x=0$. A 'generalized basis' (introduced by the authors) is a family of points $\{x_i\}$, $i \in I$ (an index set) such that there exists a family $\{\varphi_i\}$ of continuous linear functionals biorthogonal to $\{x_i\}$ ($\varphi_i(x_j)=\delta_{ij}$) and such that $\varphi_i(x)=0$ for all $i \in I$ implies $x=0$. In the latter concept, the earlier totalness and denumerability have been discarded as requirements. Theorem: If $\{x_i\}$ is a generalized basis in U then the associated family $\{\varphi_i\}$ of coefficient functionals is unique if and only if $\{x_i\}$ is total in U. Examples from analysis (Fourier series and analytic functions) are given of the various kinds of bases. If U and V are two spaces, if T is an isomorphism of U onto V, and if $\{x_i\}$ is a generalized basis in U, then obviously $\{y_i\}$, $y_i=Tx_i$, is a generalized basis in V. Each $x \in U$ gives rise for a given $\{\varphi_i\}$ to a coefficient function $\{a_i\}$, $i \in I$, $\varphi_i(x)=a_i$. Similarly for $y \in V$. Call the families of such functions $\Phi(U)$ and $\Psi(V)$. If V is isomorphic to U, then $\Phi(U)=\Psi(V)$. The converse is here proved for complete metric linear spaces. The proof generalizes a result of Arsove [Math. Ann. **135** (1958), 283–293; MR **20** #7216]. It is based upon a variation of Gel'fand's argument on the uniqueness of norm in a ring without radical.

An extension of the Paley-Wiener theorem (if $\{x_i\}$ is a total generalized basis in U and $\{y_i\}$ is in the right way and sufficiently close to $\{x_i\}$, then $\{y_i\}$ is a total generalized basis) is given for complete metric spaces. It is shown that if $\{x_i\}$ is a total generalized basis in U_n, where $\{U_n\}$ is an increasing sequence of locally convex spaces, then $\{x_i\}$ is also a total generalized basis in their inductive limit. There are two theorems (not described in this review) on bases arising from translates of certain functions on the L^2 space over a discrete abelian group. The last section deals with the continuity of the coefficient functionals. The well-known result that any basis in a Banach space is a Schauder basis has been extended to Fréchet spaces by Newns [Philos. Trans. Roy. Soc. London. Ser. A **245** (1953), 429–468; MR **14**, 968]. The authors prove two results showing that in certain spaces every weak basis of a certain type is a basis of that type (in a weak basis, the convergence of the expansion $\sum_{n=1}^{\infty} a_n x_n$ to x is assumed in the weak topology). Finally it is shown that if U is the strict inductive limit of an increasing sequence of Fréchet spaces, then every basis in U is a Schauder basis.

E. R. Lorch (New York)

5872:

Bessaga, C.; Pełczyński, A. On bases and unconditional convergence of series in Banach spaces. Studia Math. **17** (1958), 151–164.

If (x_n) is a basis for a Banach space X, denote by (f_n) the sequence in X^* biorthogonal to (x_n). After some preliminary results on block bases and equivalent bases, the authors prove theorem 3: If (x_n) is a basis for X and (y_n) is a sequence such that $\inf\|y_n\|>0$ and $f_i(y_n)\to 0$ for each i, then some subsequence of (y_n) is a basis for the (infinite-dimensional) subspace which it spans. Theorem 4: A Banach space X contains a subspace isomorphic to l which is complemented in X, if and only if X^* contains a subspace isomorphic to c_0. A number of easy corollaries follow, the most noteworthy being the known result that every infinite-dimensional Banach space contains an infinite-dimensional subspace with a basis. The notions of weak unconditional convergence (wuc) and unconditional convergence (uc) for series are introduced, and it is shown (theorem 5) that X admits a wuc series which is not uc if and only if X contains a subspace isomorphic to c_0. More corollaries follow, including the (known) result that wuc series are uc in weakly complete Banach spaces. It is shown that there exist non-weakly complete spaces in which wuc series are uc. In a final section, some of these results are extended to certain metric linear spaces.

R. R. Phelps (Berkeley, Calif.)

5873:

Bessaga, C.; Pełczyński, A. Some remarks on conjugate spaces containing subspaces isomorphic to the space c_0. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 249–250. (Russian summary, unbound insert)

This paper contains more corollaries to theorem 4 of the paper reviewed above: If X contains no complemented subspaces isomorphic to l, then wuc series in X^* are uc. If X contains a subspace isomorphic to c_0, then X^* contains a complemented subspace isomorphic to l. Every separable Banach space is a continuous linear image of X if and only if X contains a complemented subspace isomorphic to l. A conjugate space cannot contain a complemented subspace isomorphic to c_0.

R. R. Phelps (Berkeley, Calif.)

5874:

Bessaga, C.; Pełczyński, A. A generalization of results of R. C. James concerning absolute bases in Banach spaces. Studia Math. **17** (1958), 165–174.

This paper is essentially a sequel to the two reviewed above. In it, the authors prove two theorems, extending results of R. C. James [Ann. Math. (2) **52** (1950), 518–527; MR **12**, 616] on Banach spaces with an absolute basis to Banach spaces Y which can be embedded in a space with an absolute basis. Theorem 1: Every bounded subset of Y is conditionally weakly {sequentially} compact if and only if Y contains no subspace which is isomorphic to the space c_0. Some five corollaries to these (and previous) results are given, and the paper concludes with a list of unsolved problems. Samples: Does every infinite-dimensional Banach space contain an infinite-dimensional subspace with an absolute basis? Can every separable reflexive Banach space be embedded in a Banach space with an absolute basis? In a note added in proof it is pointed out

that stronger forms of the above results have been obtained by the same authors in Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 313–315 [MR **20** #3438].

R. R. Phelps (Berkeley, Calif.)

5875:

Pelczyński, A. A connection between weakly unconditional convergence and weakly completeness of Banach spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 251–253. (Russian summary, unbound insert)

A Banach space X has property (u) if for every weakly convergent sequence (x_n) in X there is a sequence (y_n) such that the series $\sum y_n$ is wuc (see #5872 above) and the sequence $x_n - \sum_{i=1}^{n} y_i$ converges weakly to 0. The following results are announced. Theorem 2: If X has property (u), then so do the subspaces of X. Theorem 3: Spaces with an absolute basis have property (u), as do weakly complete spaces (proposition 2), but $C[0, 1]$ does not have the property. The following theorems are valid in spaces X with property (u). Proposition 1: Equivalence of wuc and uc implies that X is weakly {sequentially} complete. Theorem 1: No subspace of X is isomorphic to c_0 if and only if X is weakly {sequentially} complete.

A notion of closure is then defined for subsets of a conjugate space, which is used to obtain criteria for weak completeness of conjugate spaces.

R. R. Phelps (Berkeley, Calif.)

5876:

Bessaga, C.; Pelczyński, A. Banach spaces non-isomorphic to their Cartesian squares. I. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **8** (1960), 77–80. (Russian summary, unbound insert)

In his book, *Théorie des opérations linéaires* [Monogr. Mat. Tom. I, Warsaw, 1932], Banach posed the problem of determining whether or not every infinite-dimensional Banach space is isomorphic to its Cartesian square. The authors show that a space invented by R. C. James for other purposes, and related spaces, are not isomorphic to their squares. The space X of James [Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 174–177; MR **13**, 356] is separable and isomorphic to X^{**}, and the canonical image of X in X^{**} has deficiency one.

B. Yood (Eugene, Ore.)

5877:

Semadeni, Z. Banach spaces non-isomorphic to their Cartesian squares. II. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **8** (1960), 81–84. (Russian summary, unbound insert)

Let $C(\Gamma)$ be the Banach space of all continuous real-valued functions on the compact Hausdorff space Γ where Γ is the set of all ordinals less than or equal to the first uncountable ordinal and Γ is given the order topology. The author shows that $C(\Gamma)$ is not isomorphic to its Cartesian square. Examples of infinite-dimensional Banach spaces not so isomorphic were first given by Bessaga and Pelczyński in the paper reviewed above.

B. Yood (Eugene, Ore.)

5878:

Alexiewicz, A.; Semadeni, Z. The two-norm spaces and their conjugate spaces. Studia Math. **18** (1959), 275–293.

The authors continue their study of a special class of linear topological spaces, the two-norm spaces. [See Alexiewicz, Studia Math. **14** (1953), 49–56; MR **15**, 880; and Alexiewicz and Semadeni, ibid. **17** (1958), 121–140; MR **20** #6644.] For each two-norm space $(X, \| \;\|, \| \;\|^*)$, a two-norm space $(\Xi^*, \| \;\|^*, \| \;\|)$ conjugate to the first is defined by imposing on the set Ξ^* of linear functionals on X which are continuous with respect to the norm $\| \;\|^*$ the two duals of the norms originally given in X. This allows the study of the second conjugate of a two-norm space, the canonical embedding of such a space in its second conjugate, reflexivity of two-norm spaces, and "γ-completion" of two-norm spaces.

M. M. Day (Urbana, Ill.)

5879:

Klee, Victor. Some new results on smoothness and rotundity in normed linear spaces. Math. Ann. **139**, 51–63 (1959).

Let E be a linear Hausdorff space and let F be a linear subspace of the set E of all linear functionals on E. The author has extended the usual notions of smoothness and rotundity of convex bodies to apply to more general convex sets. His new treatment yields many interesting results.

An F-hyperplane in E is a set $f^{-1}(r)$ with f in F and r real. An F-support point of a convex set C is a point p of C such that there is an f in F such that $f(x) \leq f(p)$ for all x in C and there is an x in C for which the inequality holds; then the set $f^{-1}(f(p))$ is an F-hyperplane of support to C at its F-support point p.

A point p of C is a point of F-smoothness of C provided that all F-hyperplanes of support to C at p have the same intersection with the affine extension of C. A point p of C is a point of F-rotundity of C if every F-hyperplane of support to C at p contains no other point of C. C is F-smooth [F-rotund] provided that each of its points is a point of F-smoothness [F-rotundity].

Suitable choices for F are E or E^* or EC^*, the set of linear functionals on E whose restrictions to C are continuous. If C is a convex body and $F \supseteq E^*$, F-smoothness and F-rotundity agree with the usual concepts.

The first part of this paper strengthens an isomorphism theorem for separable spaces of the reviewer [Trans. Amer. Math. Soc. **78** (1955), 516–528; MR **16**, 716]. In Proc. Amer. Math. Soc. **6** (1955), 313–318 [MR **16**, 832] the author showed that every convex closed subset C of a Banach space has an inside point; that is, a point p such that the union of the set of all those lines L through p such that p is interior to the interval $C \cap L$ is a dense subset of the affine extension of C. Here he proves a very strong approximation theorem: Suppose that C is a bounded closed subset of a separable Banach space, that 0 is an inside point of C, and that $0 < \varepsilon < 1$. Then there is a closed convex homeomorph K of C which is EC^*-smooth and E-rotund while $\varepsilon C \subset K \subset C$.

The second part considers the uniform space $\mathscr{K}$ of all non-empty compact convex subsets of a locally convex Hausdorff linear space E. 2.1 asserts that if K is an infinite-dimensional member of $\mathscr{K}$, then E^*-rotundity of K implies that the set of extreme points of K is dense in K. 2.2 asserts that when E is an infinite-dimensional Banach space, the class $\mathscr{R}$ of all E^*-rotund members of $\mathscr{K}$ is a dense G_δ in $\mathscr{K}$. It follows that (D) in a Banach space E almost every convex closed set is the closure of its set

of extreme points. 2.3 asserts that $\mathscr{S}$, the family of all E^*-smooth members of $\mathscr{X}$, is also a dense G_δ in $\mathscr{X}$.

The third part improves on some results of Köthe and the reviewer on smoothness or rotundity of quotient spaces. *M. M. Day* (Urbana, Ill.)

5880:

Gohberg, I. C.; Markus, A. S. Two theorems on the gap between subspaces of a Banach space. Uspehi Mat. Nauk **14** (1959), no. 5 (89), 135–140. (Russian)

The gap between subspaces $\mathfrak{N}$, $\mathfrak{M}$, of a Banach space $\mathfrak{B}$ is defined as

$$\tilde{\theta}(\mathfrak{N}, \mathfrak{M}) = \max[\sup_{x \in \mathfrak{S}(\mathfrak{N})} \rho(x, \mathfrak{S}(\mathfrak{M})), \sup_{y \in \mathfrak{S}(\mathfrak{M})} \rho(y, \mathfrak{S}(\mathfrak{N}))],$$

where $\mathfrak{S}(\mathfrak{A})$ is the unit sphere of $\mathfrak{A}$ and $\rho(x, \mathfrak{A})$ is the distance from x to $\mathfrak{A}$. This is a slight modification of a concept whose properties and applications are set forth by I. C. Gohberg and M. G. Kreĭn, Uspehi Mat. Nauk **12** (1957), no. 2 (74), 43–118 [Amer. Math. Soc. Transl. (2) **13** (1960), 185–264; MR **20** #3459; **22** #3984]. The modification serves to make $\tilde{\theta}$ a metric on the set $\mathfrak{R}(\mathfrak{B})$ of all (closed linear) subspaces of $\mathfrak{B}$. Theorem 1: The metric space $\mathfrak{R}(\mathfrak{B})$ is complete. Next, the minimal angle between subspaces is defined by

$$\sin \phi^{(m)}(\mathfrak{M}, \mathfrak{N}) = \inf\{|x+y| : x \in \mathfrak{M}, y \in \mathfrak{N}, \max(|x|, |y|) = 1\}$$

subject to $0 \leq \phi^{(m)} \leq \pi/2$. Lemma 1: The minimal angle between two subspaces is positive if and only if their intersection is 0 and their direct sum is closed. Theorem 2: If $\phi^{(m)}(\mathfrak{M}, \mathfrak{N}) > 0$ and the subspaces $\mathfrak{M}_1$, $\mathfrak{N}_1$ satisfy

$$\tilde{\theta}(\mathfrak{M}, \mathfrak{M}_1) + \tilde{\theta}(\mathfrak{N}, \mathfrak{N}_1) < \sin \phi^{(m)}(\mathfrak{M}, \mathfrak{N}),$$

then also $\phi^{(m)}(\mathfrak{M}_1, \mathfrak{N}_1) > 0$. If, moreover, $\mathfrak{M} + \mathfrak{N} = \mathfrak{B}$, then also $\mathfrak{M}_1 + \mathfrak{N}_1 = \mathfrak{B}$. *M. Jerison* (Lafayette, Ind.)

5881:

Zuhovickiĭ, S. I. Minimal extensions of linear functionals on the complex space $C(Q)$. Issledovaniya po sovremennym problemam teorii funkciĭ kompleksnogo peremennogo, pp. 531–534. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

This note contains proofs of results announced earlier in Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 409–422 [MR **19**, 566]. {In line 10 of the review cited, read $\bar{\phi}$ for Φ.} *E. Hewitt* (Seattle, Wash.)

5882:

Kolmogoroff, A. N.; Tichomirow, W. M. ★Arbeiten zur Informationstheorie. III. ε-Entropie und ε-Kapazität von Mengen in Funktionalräumen. Mathematische Forschungsberichte, X. VEB Deutscher Verlag der Wissenschaften, Berlin, 1960. 88 pp. DM 22.60.

Translation of Uspehi Mat. Nauk **14** (1959), no. 2 (86), 3–86 [MR **22** #2890].

5883:

Singer, Ivan. Sur les applications linéaires intégrales des espaces de fonctions continues. I. Rev. Math. Pures Appl. **4** (1959), 391–401.

Let Q be a compact space, $C(Q) = C(Q, R)$ the Banach space of all continuous real functions on Q, E a real Banach space, $C(Q, E)$ the Banach space of all continuous functions on Q with values in E. A continuous linear mapping u of $C(Q)$ into the dual space E' is "integral" (Grothendieck) if there is a (necessarily unique) continuous linear form U on $C(Q, E)$ characterized by $U(\varphi x) = \langle x, u\varphi \rangle$, $\varphi \in C(Q)$, $x \in E$. The integral norm $\|u\|_{\text{int}}$ is the norm of U in the space dual to $C(Q, E)$. Also u is "majorized" (Dieudonné) if there is a positive Radon measure μ on Q such that $\|u\varphi\| \leq \int |\varphi| d\mu$. We shall assume this is the smallest such μ (which must exist). The author shows that u is integral if and only if u is majorized and $\|u\|_{\text{int}} = \|\mu\|$. Let $\mathfrak{M}^1(Q)$ be the space dual to $C(Q)$, of all Radon measures on Q. Then u is majorized if and only if the restriction ${}^t u$ to E of the transposed mapping sends the unit ball of E onto a lattice-bounded subset of $\mathfrak{M}^1(Q)$, and then $\mu = \sup |{}^t ux|$ for $\|x\| \leq 1$. Moreover u is integral if and only if ${}^t u$ is integral from E into $\mathfrak{M}^1(Q)$, and then $\|u\|_{\text{int}} = \|{}^t u\|_{\text{int}}$. *L. Nachbin* (Waltham, Mass.)

5884:

Singer, I. Sur les applications linéaires intégrales des espaces de fonctions continues à valeurs vectorielles. Acta Math. Acad. Sci. Hungar. **11** (1960), 3–13. (Russian summary, unbound insert)

Following the notation of the preceding review, let also F be a real Banach space. A continuous linear mapping u of $C(Q, F)$ into the dual space E' is "integral" if there is a (necessarily unique) continuous linear form U on the inductive tensor product $C(Q, F) \check{\otimes} E$ characterized by $U(y \otimes x) = \langle x, uy \rangle$, $y \in C(Q, F)$, $x \in E$. The integral norm $\|u\|_{\text{int}}$ is the norm of U in the space dual to this inductive tensor product. Also u is "majorized" if there is a positive Radon measure μ on Q such that $\|uy\| \leq \int \|y(q)\| d\mu(q)$. We shall assume this is the smallest such μ (which must exist). The author shows that if u is integral then u is majorized and $\|\mu\| \leq \|u\|_{\text{int}}$; and, conversely, in order that every u which is majorized should be integral it is necessary and sufficient that the natural mapping J of the projective tensor product $F \hat{\otimes} E$ into the inductive tensor product $F \check{\otimes} E$ be a topological isomorphism between these spaces, and then $\|u\|_{\text{int}} \leq \|J^{-1}\| \cdot \|\mu\|$. Two proofs are given, one being direct and another being based on integral representations for u. *L. Nachbin* (Waltham, Mass.)

5885:

Singer, Ivan. Sur les applications linéaires majorées des espaces de fonctions continues. Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **27** (1959), 35–41.

In the notation of the preceding reviews, the author shows that u is majorized if and only if there is a (necessarily unique) continuous linear form U on $C(Q, F \hat{\otimes} E)$ characterized by $U(yx) = \langle x, uy \rangle$, $y \in C(Q, F)$, $x \in E$, where $yx \in C(Q, F \hat{\otimes} E)$ is defined by $(yx)(q) = y(q) \otimes x$. Then $\|\mu\| = \|U\|$. *L. Nachbin* (Waltham, Mass.)

5886:

Salinas, Baltasar R. Uniform approximation of a continuous function by a convex set of continuous functions. Collect. Math. **11** (1959), 175–203. (Spanish)

Let $\mathbf{C}(\mathbf{X}, \mathbf{Y})$ be the space of continuous functions h defined on a compact topological space $\mathbf{X}$ with values y in a normed vector space $\mathbf{Y}$ over the real numbers: write $\|y\|$ for the norm of y in $\mathbf{Y}$, and let $\|h\| = \sup \|y\|$ for $y = h(x)$ and $x \in \mathbf{X}$. The author considers approximations to a (fixed) $f \in \mathbf{C}(\mathbf{X}, \mathbf{Y})$ by functions ψ in a convex subset $\mathbf{K}$ of $\mathbf{C}(\mathbf{X}, \mathbf{Y})$. He denotes $\inf \|\psi(x) - f(x)\|$ for $x \in \mathbf{X}$ by $\mu(\psi)$ and the set of points x, if any, at which $\|\psi(x) - f(x)\| = \mu(\psi)$ by $F(\psi)$. The set of optimal approximations ϕ to f, that is, of functions $\phi \in \mathbf{K}$ for which $\mu(\phi) = \inf \mu(\psi)$ $(\psi \in \mathbf{K})$, is denoted by K. He generalizes results obtained by V. K. Ivanov [Mat. Sb. (N.S.) **30** (**72**) (1952), 543–558; MR **14**, 254] for the case where $\mathbf{K}$ is a $2n$-dimensional linear variety and $\mathbf{Y}$ is two-dimensional. (The corresponding results for dimensionalities n and one, respectively, are classical.) He shows (theorem 14 and corollary) that K is closed in $\mathbf{K}$, bounded, and convex, and that if $\mathbf{K}$ is closed and finite-dimensional on every point $x \in \mathbf{X}$ and is such that every bounded subset of $\mathbf{K}$ is equicontinuous, then K is compact and not empty. He supposes next that there is a real-valued function $H(t)$, continuous and increasing for real $t \geqq 0$, such that, for all y and y' in $\mathbf{Y}$,

$$\{H(\|y + \lambda y'\|) - H(\|y\|)\}/\lambda$$

tends to a limit $L(y, y')$, from above unless $y' = 0$, uniformly in every bounded subset of $\mathbf{Y} \times \mathbf{Y}$, as $\lambda \to 0$. (Thus $\frac{1}{2}L(y, y')$ is a "generalized scalar product" of y and y'.) Further, when $\phi_0 \in \mathbf{K}$ he calls a subset N of $\mathbf{X}$ a normal set with respect to ϕ_0 when for every $\phi \in \mathbf{K}$ there is an x in N for which $g(x) = L(\phi_0(x) - f(x), \phi(x) - \phi_0(x)) \geqq 0$. The set of functions g, for all $\phi \in K$, is denoted by $\mathbf{M}(\phi_0)$. He shows (theorem 19) that $\phi \in K$ if and only if $F(\phi)$ is a normal set with respect to ϕ. When $\mathbf{K}$ is an n-dimensional linear variety he shows (theorem 21) that K is not empty, that there is a set N that is a minimal normal set with respect to every $\phi \in K$ and is contained in $F = \bigcap_{\psi \in K} F(\psi)$, and also that, if K is of dimension r, if N consists of p points, and if the dimension of $\mathbf{M}(\phi_0)$ $(\phi_0 \in K)$ on every subset of $\mathbf{X}$ (or F) that consists of q points is q, then $q + 1 \leqq p \leqq n - r + 1$. (The corresponding inequalities found by Ivanov for his case are obtained when n is replaced by $2n$, q by n, and r by 0.) *H. P. Mulholland* (Exeter)

5887:

Cunningham, F., Jr. *L*-structure in *L*-spaces. Trans. Amer. Math. Soc. **95** (1960), 274–299.

Let X be a Banach space. A projection P of X onto a closed subspace is an L projection provided that for every $x \in X$, $\|x\| = \|Px\| + \|(I - P)x\|$. The L structure of X is the set of all its L projections. It is shown that the L structure of any Banach space is a complete Boolean algebra of commuting projections. The structure of this algebra, its Stone space and the measures defined over it are investigated. In particular it is shown that if X is the L space $L(E)$ of all σ-additive measures on a Boolean σ-ring E and if $\mathscr{F}$ is the L decomposition of X, then X is isometric to the space $\mathscr{N}(\mathscr{F})$ of all normal measures on $\mathscr{F}$, where a normal measure is a completely additive measure whose support in the Stone space of E is open. Another principal result is that X is an L space if and only if the L decomposition of X is maximal as an Abelian family of projections of unit norm. Since it is known that any L space can be represented as $L(E)$ for a suitable σ-ring E, this yields a representation of any L space as a space $\mathscr{N}(\mathscr{F})$, $\mathscr{F}$ being unique up to isomorphism.

R. E. Fullerton (College Park, Md.)

5888:

Lighthill, M. J. ★Introduction to Fourier analysis and generalised functions. Cambridge University Press, New York, 1960. viii + 79 pp. \$1.95.

Paperback reprinting of the first (1958) edition, reviewed in MR **19**, 1066.

5889:

Gelfand, I. M.; Schilow, G. E. ★Verallgemeinerte Funktionen (Distributionen). I: Verallgemeinerte Funktionen und das Rechnen mit ihnen. Hochschulbücher für Mathematik, Bd. 47. VEB Deutscher Verlag der Wissenschaften, Berlin, 1960. 364 pp. DM 37.00.

This translation is based on the second Russian edition [Gosudarstv. Izdat. Fiz. Mat. Lit., Moscow, 1959]. For a review of the first edition (1958) see MR **20** #4182.

5890:

Laugwitz, Detlef. Eine Einführung der δ-Funktionen. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1959**, 41–59 (1960).

Generalized Ω-numbers (arbitrary sequences of rational numbers) are defined, and operations, functions, limits and δ-functions defined as in a previous paper by Schmieden and the author in Math. Z. **69** (1958), 1–39 [MR **20** #2404]. These ideas are applied to deduce formulae concerning Green's functions and fundamental solutions of differential equations. It is shown that products of the delta functions may be defined in this system. It does not appear that the delta functions involved necessarily have derivatives; and it is not clear what bearing the derivation of formulae in the calculus here presented has on the proof of the validity of corresponding formulae in classical analysis (or distribution theory).

J. L. B. Cooper (Cardiff)

5891:

Sebastião e Silva, José. Sur le calcul symbolique des opérateurs différentiels à coefficients variables. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **27** (1959), 42–47, 118–122.

Let L_0 be the vector space of all complex functions on the real axis $\mathbf{R}$ locally integrable and vanishing for $x < 0$. Call C_0^* the vector subspace of L_0 of all absolutely continuous complex functions on $\mathbf{R}$ vanishing for $x < 0$. The differential operator $\mathfrak{D}u(x) \equiv a(x)u'(x) + b(x)u(x)$, where $a(x)$ and $b(x)$ are locally integrable and $a(x)$ and $1/a(x)$ are locally bounded, maps C_0^* isomorphically onto L_0. There is an extension $\bar{L}_0$ of L_0 and an extension $\mathfrak{D}$ of the differential operator mapping $\bar{L}_0$ isomorphically onto $\bar{L}_0$ so that $\bar{L}_0 = \bigcup_m \mathfrak{D}^m(L_0)$, such extensions being essentially unique. The elements of $\bar{L}_0$ are called "paradistributions" vanishing on the negative axis associated to $\mathfrak{D}$. When $a(x) \equiv 1$, $b(x) \equiv 0$, these paradistributions become the distributions of finite order on $\mathbf{R}$ vanishing on the negative axis. The space $\bar{L}_0$ is endowed with a natural topology, obtained as a projective limit in the Dieudonné-Schwartz style, in whose definition $\mathfrak{D}$ plays a rôle. The completion of $\bar{L}_0$ is the analogue of the space of distributions of

arbitrary order on **R** vanishing on the negative axis. Let $\mathfrak{A}$ be the topological algebra of all continuous linear endomorphisms of $\bar{L}_0$ endowed with the topology of either simple or bounded convergence. $\mathfrak{D}-\lambda$ is invertible for all complex λ. Under the condition that $\mathfrak{D}-\lambda$ is bounded on every left half plane $\operatorname{Re}\lambda \leq \alpha$, which is the case if $a(x)$ is positive, there exists an operational calculus

$$\varphi(\mathfrak{D}) = \frac{1}{2\pi i}\int_{\alpha-\infty i}^{\alpha+\infty i} \frac{\varphi(\lambda)}{\mathfrak{D}-\lambda}\,d\lambda,$$

where the integral is understood in a suitable generalized sense, for every function belonging to the topological algebra $\mathfrak{U}_\infty$ of all complex functions φ such that $\varphi(z)/z^k$ is continuous and bounded for $\operatorname{Re} z \geq k$ and analytic for $\operatorname{Re} z > k$ for some $k=0, 1, \cdots$, $\mathfrak{U}_\infty$ being endowed with its natural inductive limit topology. Then $\varphi \to \varphi(\mathfrak{D})$ is the unique continuous homomorphism of $\mathfrak{U}_\infty$ into $\mathfrak{A}$ mapping z into $\mathfrak{D}$ and 1 into I. The case of second order operators and the pertinent modifications are also considered. Several possible variations are briefly pointed out.

L. Nachbin (Waltham, Mass.)

5892:

Schaefer, H. H. On the Fredholm alternative in locally convex linear spaces. Studia Math. **18** (1959), 229–245.

Let E and F be linear spaces with a locally convex topology and T a homomorphism of E into F. Specific topologies such as the weak, Mackey and strong topologies also appear. A number of notions for T, such as "σ transformation", "Fredholm transformation", "Riesz transformation" and "bounded transformation" are considered, and for $F=E$, spectral characteristics. T is a σ transformation if its null space $N(T)$ is finite dimensional and its range $T(E)$ has a finite dimensional quotient space $Q(E)$. A weakly continuous T is a Fredholm transformation if $T(E)$ is the set orthogonal to the set of zeros of T', the adjoint of T, or $T'(E)$ is the set orthogonal to $N(T)$ and $N(T)$ and $N(T')$ have the same finite dimension. If $E=F$, a weakly continuous T is a Riesz transformation if $N(T^n)$ is fixed and finite dimensional for $n \geq n_0$, and $T^m(E)$ is closed and fixed for $m \geq m_0$ and $Q(T^m)$ is finite dimensional. T is bounded if it takes some open set into a bounded one. The complex numbers λ for which $\lambda I - T$ is a Fredholm transformation constitute the Fredholm domain of T and a similar definition is given for the Riesz domain. Structural characteristics of σ transformations relative to various topologies are discussed and also the structure of the other types of transformations. If T is bounded and $E=F$ is sequentially closed, then the spectrum of T is shown to be closed, the Fredholm and Riesz domains are shown to be open, and the component relations of the latter are given.

F. J. Murray (New York)

5893:

Schaefer, Helmut. Some spectral properties of positive linear operators. Pacific J. Math. **10** (1960), 1009–1019.

Let E be a partially ordered Banach space with positive cone K, T a positive operator (bounded linear operator in E that maps K into itself), r the spectral radius of T, and R_λ its resolvent operator. The author proves that certain topological conditions on K imply properties of the spectrum, spectral radius, and resolvent of an arbitrary positive operator. Also, by imposing conditions on the positive operator stronger conclusions are obtained. A positive operator T is said to be quasi-interior if there exists $\lambda > r$ such that $TR_\lambda x$ is a quasi-interior point of K for every non-zero point x of K. Among other results it is proved that if T is quasi-interior and r is a pole of R_λ, then (1) $r>0$ and r is a simple pole of R_λ, and (2) every characteristic vector of T in K corresponding to r is quasi-interior to K. Certain additional conditions are given which imply that the nullspace of $rI-T$ has dimension one.

F. F. Bonsall (Newcastle-upon-Tyne)

5894:

Halberg, Charles J. A., Jr.; Kramer, Vernon A. A generalization of the trace concept. Duke Math. J. **27** (1960), 607–617.

Let T be a self-adjoint transformation on Hilbert space which is semi-bounded from below and such that $T_\sigma^{-1} = (T+\sigma I)^{-1}$ is in the trace class for some positive σ. The eigenvalues of T, ordered according to magnitude and with the proper multiplicity, are $\{\mu_n\}$, with $\{e_n\}$ as a corresponding orthonormal system of eigenelements. The following theorems are proved. (i) Let V be bounded and such that $T+V$ has the sequence of real eigenvalues $\{\lambda_n\}$ with $\sum(\lambda_n-\mu_n)$ convergent. Then

$$\sum(\lambda_n-\mu_n) = \lim \rho^2 S\{T_\rho^{-1}VT_\rho^{-1}\}$$

as $\rho\to\infty$, where S denotes the trace. (ii) If, in addition, $\sum(Ve_n, e_n)$ converges, then $\sum(\lambda_n-\mu_n) = \sum(Ve_n, e_n)$.

These results are applied to the case that, on the interval $[0, \pi]$, T is either $-d^2/dx^2$ or d^4/dx^4 with any self-adjoint boundary conditions, and V is multiplication by $q(x)$, where $q(x)$ is differentiable and satisfies $\int_0^\pi q(x)dx=0$. A well-known result of I. M. Gel'fand and B. M. Levitan [Dokl. Akad. Nauk SSSR **88** (1953), 593–596; MR **15**, 33], asserting that for the second order problem with boundary conditions $y'(0)=y'(\pi)=0$ one has

$$\sum(\lambda_n-\mu_n) = [q(0)+q(\pi)]/4,$$

is thereby considerably extended.

A. C. Zaanen (Pasadena, Calif.)

5895:

Foiaş, Ciprian. Décompositions intégrales des familles spectrales et semi-spectrales en opérateurs qui sortent de l'espace hilbertien. Acta Sci. Math. Szeged **20** (1959), 117–155.

This paper is concerned with general spectral resolution for operators in Hilbert space. The main point of it is, roughly speaking, that it puts in a general setting, and extends, various results on generalized "eigenfunction" expansions, in the sense of Mautner [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 49–53; MR **14**, 659], and later Browder [Amer. J. Math. **80** (1958), 365–381; MR **20** #1064], and Gel'fand and Kostyučenko [Dokl. Akad. Nauk SSSR **103** (1955), 349–352; MR **17**, 388]. The context is rather general, and in the special case of normal operators a complete unitary invariant is found.

Here are some of the details. Let H be a separable Hilbert space; T a locally compact space; B a countably additive (in the weak topology) positive operator-valued (operators on H) function on Borel sets in T, such that $0\leq B(\sigma)\leq I$ (this is essentially an operator measure; see Schreiber [Trans. Amer. Math. Soc. **87** (1958), 108–118; MR **20** #6040], Sz.-Nagy [same Acta **15** (1953), 87–92; MR **15**, 326]); E a dense subspace of H with a norm $\|\ \|_1$

such that its closure E_1 is separable and is nuclearly imbedded in H (for example, the finite linear combinations of a basis is such an E).

The first theorem is that if B is absolutely continuous with respect to a regular (numerical) measure μ (and such measures are shown to exist in a natural way) then there exists a function $x(t)$ on T whose values are continuous linear maps $E_1 \to E_1^*$ (the Banach space dual of E_1) such that $\langle x(t)f, f\rangle \geq 0$ for $f \in E_1$ and

$$(B(\sigma)f, g) = \int_\sigma \langle x(t)f, g\rangle \, d\mu(t)$$

(here $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ are inner products in H and E_1, E_1^* respectively). There is also a converse to this Radon-Nikodym theorem (which is how it is proved), to the effect that given x such that $\|x\|_1$ is integrable and $x \geq 0$ (as above) then there is a B satisfying the above relation, and x is determined (a.e.μ) uniquely by B. As for E, the result is that the rank of $x(t)$ is (a.e.μ) the same for all E of the specified type. It will be seen (compare the cited works) that $x(t)$ is essentially the generalized eigenvector (function, distribution) corresponding to t in case T is the spectrum of an operator with spectral measure B.

The next part of the paper analyses the structure of $x(t)$, and the main result is a Fourier-like expansion for $x(t)f$ ($f \in E_1$) in terms of E_1^*-valued functions on T. From these considerations generalized eigenfunction expansions are obtained. We omit details.

The last part of the paper takes up the case where B is a spectral measure (projection-valued, and $B(T)=I$). Here it is proved that B has multiplicity $\leq n$ if and only if the rank of $x(t)$ is (a.e.μ) $\leq n$; and if Ω_n = the set of t such that $x(t)$ has rank exactly n and $\mu(\Omega_n) \neq 0$, then B has multiplicity exactly n (in these statements $n = 0, 1, 2, \cdots$). Finally it is shown that if B' is another spectral measure, absolutely continuous with respect to μ, and x' is the corresponding function, then B and B' are unitarily equivalent if and only if x and x' have the same rank (a.e.μ). Thus x gives a complete unitary invariant for B, or equivalently, for the operator $A = \int t\,dB(t)$, when T is the spectrum of A.

M. Schreiber (Ithaca, N.Y.)

5896:

Foiaş, Ciprian. Sur la décomposition intégrale des familles semi-spectrales en opérateurs qui sortent de l'espace de Hilbert. C. R. Acad. Sci. Paris **248** (1959), 904–906.

This paper is a precise summary of the same author's paper reviewed immediately above.

M. Schreiber (Ithaca, N.Y.)

5897:

Kolomý, Josef. An approximate solution for a system of functional equations by Galerkin's method. Apl. Mat. **5** (1960), 296–304. (Czech. Russian and English summaries)

Let H denote a complete separable Hilbert space, A and B two symmetric and positive-definite operators with the domains D_A, D_B, dense in H, and K, L linear operators with $D_K \supset D_B$, $D_L \supset D_A$. The author deals with the system $Au_1 + Ku_2 = f_1$, $Bu_2 + Lu_1 = f_2$ ($f_1, f_2 \in H$), which he transforms to one operator equation and solves by means of Galerkin's method on applying Mihlin's theorems [*Variacionnye metody v matematičeskoĭ fizike*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957; MR **22** #1981; pp. 392–399]. Two examples are given.

M. Zlámal (Brno)

5898:

Haimovici, Adolf. Sur la méthode de Fourier pour certaines équations dans un espace de Banach. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) **5** (1959), 23–32. (Romanian and Russian summaries)

Let Ω be a Banach algebra with identity containing two subalgebras Ω_1, Ω_2, such that $\Omega_1 \cap \Omega_2$ consists of scalar multiples of the identity and $x_1x_2 = x_2x_1$ for $x_r \in \Omega_r$. Let $A_r : \Omega \to \Omega$ be linear operators such that $A_r\Omega_r \subseteq \Omega_r$, $A_1x_2x_1 = x_2A_1x_1$, $A_2x_1x_2 = x_1A_2x_2$ and let $A = A_1 - A_2$. By an obvious adaptation of the method of separation of variables in differential equations, the author studies, under complicated hypotheses, solutions of $Ax = 0$ of the form $x = x_1x_2$, $x_r \in \Omega_r$, and solutions that are limits of linear combinations of such solutions. Several mistakes occur, of which the most serious is the requirement (axiom I_2) that Ω_1, Ω_2 be normed division algebras. The paper acquires some significance if this is weakened to the condition that factorization of $x = x_1x_2$, $x_r \in \Omega_r$, is unique up to scalar multiplication.

D. A. Edwards (Newcastle upon Tyne)

5899:

Balakrishnan, A. V. Fractional powers of closed operators and the semigroups generated by them. Pacific J. Math. **10** (1960), 419–437.

Let A be a closed linear operator with domain and range in a Banach space X. Suppose each $\lambda > 0$ is in the resolvent set of A and

$$(\mathrm{H}_0) \qquad \|\lambda R(\lambda, A)\| < M < \infty, \quad \lambda > 0.$$

For $0 < \operatorname{Re} \alpha < 1$ define for each $x \in D(A)$

$$J^\alpha x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1} R(\lambda, A)(-A)x \, d\lambda,$$

and for $0 < \operatorname{Re} \alpha < 2$ define for each $x \in D(A^2)$

$$J^\alpha x = \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^\infty \lambda^{\alpha-1} \left[R(\lambda, A) - \frac{\lambda}{1+\lambda^2} \right] (-A)x \, d\lambda + [\sin \pi\alpha/2](-A)x.$$

In general for $n-1 < \operatorname{Re} \alpha < n$ and $x \in D(A^n)$ define $J^\alpha x = J^{\alpha-n+1}(-A)^{n-1}x$, and these definitions are consistent. The principal value of λ^α is taken so that $\lambda^\alpha \geq 0$ for $\alpha > 0$.

The author shows that J^α has a closure and defines $(-A)^\alpha$ = closure of J^α. Further $(-A)^{\alpha+\beta} = [(-A)^\alpha(-A)^\beta]$, where the right hand side of this equation is the closure of the product of the given operators. If $D(A)$ is dense in X then $\sigma[(-A)^\alpha] = [\sigma(-A)]^\alpha$, where σ denotes the spectrum.

If $D(A)$ is dense and A satisfies (H_0), then for $0 < \alpha \leq 1/2$, $-(-A)^\alpha$ generates a semigroup $S_\alpha(t)$ which is strongly continuous for $t \geq 0$ and uniformly continuous for $t > 0$. An application is given to an abstract Cauchy problem of the form $d^2u(t)/dt^2 + Au(t) = 0$.

A. Devinatz (Princeton, N.J.)

5900a:

Dixmier, Jacques. Sur les représentations unitaires des groupes de Lie nilpotents. V. Bull. Soc. Math. France **87** (1959), 65–79.

5900b:

Dixmier, Jacques. Sur les représentations unitaires des groupes de Lie nilpotents. VI. Canad. J. Math. **12** (1960), 324–352.

The author continues his investigation of the unitary representations of nilpotent Lie groups [Parts I–IV, Amer. J. Math. **81** (1959), 160–170; same Bull. **85** (1957), 325–388; same J. **10** (1958), 321–348; **11** (1959), 321–344; MR **21** #2705; **20** #1928, 1929; **21** #5693]. Let Γ be a connected nilpotent Lie group. In part V the author shows that the characters of irreducible unitary representations of Γ are distributions, which are furthermore tempered distributions if Γ is simply connected. It follows that Γ has a smooth dual in the sense of Mackey [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 537–545; MR **11**, 158]. Let $\bar{\Gamma}$ be the set of equivalence classes of irreducible representations of Γ. A topology for $\bar{\Gamma}$ has been introduced by Godement in Trans. Amer. Math. Soc. **63** (1948), 1–84 [MR **9**, 327]. In part III of this series the author has identified the set $\bar{\Gamma}$ for all simply connected nilpotent Lie groups Γ of dimension $\leqq 5$. In part VI, by explicit calculation, he determines the topology in six of these instances and obtains a partial result for a seventh. He also identifies the characters of two of these groups.

K. deLeeuw (Princeton, N.J.)

5901:

Dixmier, Jacques. Sur les représentations de certains groupes orthogonaux. C. R. Acad. Sci. Paris **250** (1960), 3263–3265.

Let SO(n) [resp. SO$'$(n)] denote the connected component of the group of all automorphisms of the vector space R^n ($n \geqq 3$) which preserve the form $x_1^2+x_2^2+\cdots+x_n^2$ [resp. $x_1^2+x_2^2+\cdots+x_{n-1}^2-x_n^2$]. Then SO($n$) $\cap$ SO$'$(n) is a maximal compact subgroup of SO$'$(n) isomorphic to SO($n-1$). The author shows (theorem 2) that the restriction to SO(n) $\cap$ SO$'$(n) of every completely irreducible Banach space representation of SO$'$(n) is free of multiplicities. Theorem 1, used in the proof of theorem 2, is an extension to not necessarily complex semi-simple Lie groups of a general theorem of Godement [Trans. Amer. Math. Soc. **73** (1952), 496–556; MR **14**, 620; theorem 3] about restricting irreducible representations to maximal compact subgroups. He remarks that his result can be used to rigorize the study of the representations of the De Sitter group given by Thomas [Ann. of Math. (2) **42** (1941), 113–126; MR **2**, 216].

G. W. Mackey (Cambridge, Mass.)

5902:

Dieudonné, Jean. Sur le produit de composition. II. J. Math. Pures Appl. (9) **39** (1960), 275–292.

For notations see part I of same title in Compositio Math. **12** (1954), 17–34 [MR **16**, 265]. If Φ_k is one of the function classes, let $\mathcal{T}_k$ be the corresponding topology $\sigma(\mathfrak{M}_1(G), \Phi_k)$ (weak topology on $\mathfrak{M}_1$ regarded as the dual of $\bar{\Phi}_0$). Then for $A, B \subset \mathfrak{M}_1(G)$, $A*B$ is relatively compact $\mathcal{T}_{k_1}$ if A [resp. B] are relatively compact $\mathcal{T}_{k_2}$ [resp. $\mathcal{T}_{k_3}$] for a variety of triples (k_1, k_2, k_3). Variants of this theme are also proved.

For $1 \leqq p \leqq \infty$, μ a bounded real measure on G, let $\gamma_{\mu,p}$ be the endomorphism of $L^p(G)$ given by $\gamma_{\mu,p}: f \to \mu * f$. J. Wendel [Pacific J. Math. **2** (1952), 251–261; MR **14**, 246] showed $\|\gamma_{\mu,1}\| = \|\mu\|$. Since $\|\check{\mu}\| = \|\mu\|$, we see $\|\gamma_{\mu,p}\| = \|\gamma_{\check{\mu},q}\|$, $1/p+1/q=1$. If G, μ are unrestricted, e.g., $G=(0, 1, 2)$, then in general $\|\gamma_{\mu,p}\| < \|\mu\|$ if $1<p<\infty$. If μ is positive and G is abelian, $\|\gamma_{\mu,p}\| = \|\mu\|$, $1 \leqq p \leqq \infty$. If G is abelian, G enjoys property P_p: For every compact $K \subset G$, $\varepsilon > 0$, there is $f \in L^p(G)$ such that (1) $f \geqq 0$; (2) $N_p(f)=1$; (3) $N_p(f_y - f) \leqq \varepsilon$ for $y \in K$ [H. Reiter, Trans. Amer. Math. Soc. **73** (1952), 401–427; MR **14**, 465]. If G enjoys P_p, then $\|\gamma_{\mu,p}\| = \|\mu\|$ for positive μ. Elementary results regarding preservation of P_p under products, quotients, etc., are given. An important theorem is: If G is a connected nilpotent Lie group, G enjoys P_p, $p \geqq 1$.

The paper ends with a discussion of the circumstances under which $\gamma_{\mu,p}$ is sur-, bi-, injective. For example, if G is abelian, $p=2$ and F is the Fourier transform of μ, then $\gamma_{\mu,p}$ is bijective if and only if there is a constant $c>0$ such that $|F(\xi)| \geqq c$ for all $\xi \in G$.

B. R. Gelbaum (Princeton, N.J.)

5903:

Rickart, Charles E. ★General theory of Banach algebras. The University Series in Higher Mathematics. D. van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960. xi+394 pp. $10.50.

In the two decades since Gel'fand laid the foundation for Banach algebras in his now classical 1941 paper [Mat. Sb. (N.S.) **9 (51)** (1941), 3–24; MR **3**, 51], a rapid growth of interest has occurred. Indeed, the bibliography of the book under review lists in 49 pages 775 items until 1959. "Standing as it were, between analysis and algebra", the author writes in his preface, "the theory of Banach algebras has developed roughly along two main lines representing respectively the analytic and algebraic influences. The analytic emphasis has been on the study of certain special Banach algebras, along with some generalizations of these algebras, and on extending certain properties of function theory and harmonic analysis to the more general situation offered by Banach algebras. On the other hand, the algebraic emphasis has naturally been on various aspects of structure theory. Of great importance here has been the growing interest of algebraists in algebra without finiteness restrictions. This development, which has been much stimulated by the study of Banach algebras, has supplied important new algebraic methods which are profitably applied to Banach algebras.

"It becomes increasingly evident that, in spite of the deep and continuous influence of analysis on the theory of Banach algebras, the essence of the subject as an independent discipline is to be found in its algebraic development. Therefore, as the title indicates, this book is devoted primarily to an exposition of the general theory. Needless to say, an algebraic point of view has dominated strongly the selection and organization of materials." In this aim, the present book differs greatly from M. A. Naĭmark's *Normirovannye kol'ca* [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956; MR **19**, 870], Loomis' *An introduction to abstract harmonic analysis* [Van Nostrand, New York, 1953; MR **14**, 883] and other hitherto published books which are concerned more or less with Banach algebras. Also the author is particularly well qualified to realize this aim. Hence, it is obvious that the present book occupies a unique position in the literature on abstract analysis.

The main body of the book is divided into four chapters and an appendix. Each chapter begins with its introduction and is divided into several sections. Within each

section, definitions, theorems and corollaries are numbered enclosed in parentheses ((x, y, z) means zth item in § y of chapter x). At the ends of most sections, remarks are directed to the literature involving related results. The bibliography lists almost all papers until 1959. The list of symbols contains 168 items carefully classified. The index contains 641 items (main items are cross-indexed as possible). The book assumes the readers' familiarity with the elements of Banach and Hilbert spaces, whereas the terms of general ring theory are explained briefly.

Now, we shall discuss briefly the contents chapter by chapter.

Chapter I (40 pages, 60 items enclosed in parentheses) begins with the definition in § 1. The regular representation is discussed in § 2. {Comparing with a recent announcement of P. C. Shields [Bull. Amer. Math. Soc. **65** (1959), 267–269; MR **21** #5917], it seems that the inspection of the conjugate representations of the regular representation with their strong operator topology will add some information.} The complexification of real algebras occupies § 3. After these preparatory sections, the essential discussion begins in § 4 with the calculation of the spectral radius. The existence of $\nu(x)=\lim_n\|x^n\|^{1/n}$ and the subadditivity of $\nu(x)$ on commutable elements are proved in (1.4.1) by an improvement of Riesz-Sz. Nagy's method (which is a typical example of several improvements of the author employed in the book). The circle operation $x\circ y=x+y-xy$ is introduced and quasi-regularity is discussed also in § 4. With a contribution due to Quigley, § 5 contains discussions on topological divisors of zero. § 6 contains the definition of the spectrum of an element, where the author simply decides that 0 belongs to the spectrum of each element when the algebra has no identity. The chapter ends in § 7 with the Mazur-Gel'fand Theorem (1.7.1). {The reviewer wishes to add Ono's note [see review #5904] to the list of elementary proofs of (1.7.1).}

Chapter II (67 pages, 128 items) is the general theory in the "general theory". Beginning with the discussion of ideals and the difference algebras in § 1, via a discussion of representations in § 2, the author introduces the radical of an algebra as the Jacobson radical in § 3. The radical is characterized in (2.3.5). § 4 studies primitive algebras. § 5 contains theorems on uniqueness of norm which are mainly due to the author himself, and ends with the fundamental isomorphism theorem (2.5.19). The structure space π and the strong structure space Ξ are introduced in (2.6.3) as the space of all primitive ideals and all maximal modular ideals respectively with the hull-kernel topology. § 7 is a description of the completely regular algebras of Wilcox which generalizes the "regular" rings of Šilov. (2.7.1) defines them when (i) the strong structure space is a Hausdorff space, and (ii) each point has a neighbourhood V such that the kernel of V is modular (regular in the sense of Segal). § 8 contains the annihilator algebras of Bonsall and Goldie, a generalization of the dual algebras of Kaplansky. Throughout these two chapters, it is remarkable that (1) the author employs the technical terms of Jacobson's *Structure of rings* [Amer. Math. Soc., Providence, R.I., 1956; MR **18**, 373] when possible, and (2) the author gives a complete description of algebras without identity element.

Chapter III (70 pages, 109 items) is devoted to commutative Banach algebras. Chapter III can be read, as remarked in the preface, by a reader who is interested only in the commutative case and passes directly from chapter I to III. The author uses the term "carrier space" for the space of all maximal modular ideals with the Gel'fand topology (called "$\mathfrak{A}$-topology" by the author). The Čech compactification (3.2.11) and Stone-Weierstrass Theorem (3.2.12) are the main items of § 2. The existence of the Šilov boundary of a locally compact space Ω relative to a complex subalgebra of all continuous functions vanishing at infinity is proved in (3.3.1). § 3 contains also other interesting properties of the Šilov boundary due to Šilov and Holladay. § 4 contains realizations of the carrier spaces by certain concrete sets. The main problem of § 5 is to give a partial generalization of the Cauchy integral formula for holomorphic functions of many variables due to Arens and Hervé. After a brief discussion on the direct-sum decomposition in § 6, the chapter ends in § 7 with an exposition of complete regular algebras due to Šilov.

Chapter IV (99 pages, 159 items) is the longest. It is concerned with Banach algebras with involution. Beginning in § 1 with the elementary properties of *-algebras, through Gel'fand and Naĭmark's characterization (4.2.2) of the algebra of all complex-valued continuous functions defined on a locally compact space and vanishing at infinity, the author leads us to the solution by Fukamiya, Kelley, Vaught, Kaplansky and the author, of Gel'fand and Naĭmark's conjecture (4.8.11), which states that an abstract B^*-algebra is representable isomorphically by a (uniformly closed) C^*-algebra of operators defined on a Hilbert space. §§ 3–4 are devoted to *-representations, § 5 is devoted to positive linear functionals, and they are combined in § 6. § 6 contains also a proof of Kreĭn's extension theorem of positive linear functionals (p. 227). In § 7, the symmetry of the involution is introduced and discussed. § 8 contains the proof of the main theorem and Bohnenblust and Karlin's (4.8.16) which characterizes a positive linear functional by $\|F\|=F(1)$ when the identity exists. It is noteworthy, that the Bohnenblust-Karlin theorem implies conversely the Gel'fand-Naĭmark theorem. Kadison's theorem (4.9.10), which states that a topologically irreducible representation of a B^*-algebra is strictly irreducible, is proved in § 9. Banach *-algebras with minimal ideals are discussed in § 10. Finally, in the same § 10, an Ambrose H^*-algebra is discussed as a Banach algebra which admits a perfect generalization of the classical Wedderburn structure theorems.

It is unfortunate for the readers that because of limitations of both time and space the author gave up his original plan to place the theory of rings of operators into the general theory of Banach algebras. Also, the reviewer regrets that the author omitted the following topics: (1) States and the state space due to Segal, Kadison and others; a characterization of "pure states" among states due to Tate [Comm. Pure Appl. Math. **4** (1951), 31–32; MR **13**, 361] in the commutative case; (2) Kadison's Schwarz inequality and its elegant extension due to Sz.-Nagy [Acta Math. Acad. Sci. Hungar. **3** (1952), 285–293; MR **14**, 1096]; and (3) Turumaru's direct (tensor) product of B^*-algebras which is algebraic in itself and has some algebraic properties. The reviewer does not agree with the author's avoidance of the term "state".

The appendix (56 pages) is devoted to presenting various examples which are motivated by and illustrate the general theory in the preceding chapters. It is divided into three parts: algebras of operators, algebras of functions and group algebras, which are introduced and described

briefly. The appendix is enriched by three contributions of Kakutani (p. 280, p. 282 and p. 294) which have not been published before.

It is naturally impossible for 400 pages to be written without minor faults. A few trivial faults will be listed here. "Kenkichi" (p. 354) is the first name of Iwasawa, whence his [1] is listed twice. R. B. Smith [1] cited in p. 260 is skipped in the bibliography. The index indicates that p. 211 contains the definition of "centralizer", whereas no definition is given in p. 211.

Summing up, the book is an important addition to the mathematical literature and is recommended to anyone interested in the theory of Banach algebras. Especially for the complete description of algebras without identity element, the book will be used by the specialists as a standard reference book in the future.

M. Nakamura (Asiya)

5904:

Ono, Tamio. Elementary proof of the basic theorem of normed rings. Sûgaku **9** (1957/58), 236. (Japanese)

The author gives a proof of Gel'fand's formula of the spectral radius [cf. M. A. Naĭmark, *Normirovannye kol'ca*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956; translation, *Normed rings*, Noordhoff, Groningen, 1959; MR **19**, 870; **22** #1824; and C. E. Rickart, *General theory of Banach algebras*; #5903 above; p. 30] using roots of unity instead of analytic function theory. Independently of this note, a similar proof has been given by Rickart [Michigan Math. J. **5** (1958), 75–78; MR **20** #4786].

Z. Takeda (Hitachi)

5905:

Ono, Tamio. Note on a B^*-algebra. J. Math. Soc. Japan **11** (1959), 146–158.

A *-algebra R is a complex Banach algebra (not necessarily having an identity element) with a mapping $x \to x^*$ which is a conjugate linear anti-isomorphism of period two. If moreover $\|x^*x\| = \|x^*\|\,\|x\|$, call R a B'^*-algebra. A B'^*-algebra is called a B^*-algebra provided $\|x^*\| = \|x\|$. A *-algebra is called C-symmetric if every maximal commutative *-subalgebra is a B^*-algebra. Finally a C^*-algebra is a uniformly closed self-adjoint algebra of bounded operators on a complex Hilbert space.

This paper presents proofs that (A) a C-symmetric algebra is homeomorphic and *-isomorphic to a C^*-algebra, and (B) a B'^*-algebra is a B^*-algebra. The author also has a number of interesting corollaries and partial results. Unfortunately, there seem to be errors in the proofs of both of the main results.

{Reviewer's notes: Gel'fand and Naĭmark [Mat. Sb. (N.S.) **12** (**54**) (1943), 197–219; MR **5**, 147] conjectured that a B'^*-algebra is isometric and *-isomorphic to a C^*-algebra. Since it is known that this is true for a B^*-algebra [e.g., Theorem 4.8.11 of Rickart, *General theory of Banach algebras*; #5903 above], a correct proof of (B) would verify this conjecture. An independent proof of this conjecture (assuming that R has an identity element) has been published by Glimm and Kadison [see following review].}

J. A. Schatz (Albuquerque, N.M.)

5906:

Glimm, James G.; Kadison, Richard V. Unitary operators in C^*-algebras. Pacific J. Math. **10** (1960), 547–556.

The authors derive several in certain respects definitive results on C^*-algebras (with unit). They proceed from the striking if quite simple result of Kadison on irreducible C^*-algebras [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 273–276; MR **19**, 47], via a key theorem stating roughly that the unitary elements of an irreducible (concrete) C^*-algebra with unit are suitably dense in the set of all unitary operators on the representation space. Specifically, it is shown that if V is a unitary operator on the irreducible representation space $\mathscr{H}$ for the C^*-algebra $\mathscr{A}$, which takes the vector x_k into the vector y_k, k ranging over a finite index set, there is a unitary operator U in $\mathscr{A}$ with the same property.

It follows that (1) two pure states of a C^*-algebra are unitarily equivalent provided the corresponding representations are so; (2) the *-operation is isometric in B^*-algebras; (3) two pure states ρ and σ are unitarily equivalent provided $\|\rho-\sigma\| < 2$.

I. E. Segal (Cambridge, Mass.)

5907:

Šilov, G. E. Analytic functions in a normed ring. Uspehi Mat. Nauk **15** (1960), no. 3 (93), 181–183. (Russian)

Let A be a commutative Banach algebra with unit, and let $\Delta = \mathrm{Hom}(A, \mathbf{C})$. A function $f \in \mathscr{C}(\Delta, \mathbf{C})$ is (here) called 'locally analytic with respect to A' if for each $\zeta \in \Delta$ there are $a_1, \cdots, a_n$ in A such that $\hat{a}_i(\zeta) \equiv \zeta(a_i) = 0$ and $f = f(\zeta) + \sum \lambda_{i_1,\dots,i_n} \hat{a}_1^{i_1} \cdots \hat{a}_n^{i_n}$ in some neighborhood of ζ, where $\sum |\lambda_{i_1,\dots,i_n}| < \infty$. Does it follow that $f = \hat{a}$ for some $a \in A$? This has long ago been recognized as a wonderfully definitive generalization of the Šilov-Waelbroek-Calderón-Arens functional-operational calculus theory. Now the author professes to sketch a proof that f does have the form $\hat{a}$, for some $a \in A$. Suppose in fact that $f = \hat{a}_1$ on one neighborhood U_1 and $f = \hat{a}_2$ on U_2 where $U_1 \cup U_2 = \Delta$. The author's argument suggests defining a function φ on the joint spectrum $\sigma(a_1, a_2) \subset \mathbf{C}^2$ by letting $\varphi(z_1, z_2) = z_i$ on the image of U_i $(i = 1, 2,)$ and extending this holomorphically to a neighborhood of $\sigma(a_1, a_2)$, so that the ŠWCA method may be applied. The reviewer does not see that such an extension of φ is possible. The most that can be hoped for is an approximation of φ by polynomials on $\sigma(a_1, a_2)$, but hardly in a neighborhood (see Rickart, *General theory of Banach algebras* [#5903] for references).

R. Arens (Los Angeles, Calif.)

5908:

Bear, H. S. A strong maximum modulus theorem for maximal function algebras. Trans. Amer. Math. Soc. **92** (1959), 464–469.

The author studies a certain generalization of the classical "strong" maximum principle of function theory which says that a non-constant analytic function in the disk has at each interior point a modulus strictly smaller than the maximum modulus taken over the boundary. Let X be a compact Hausdorff space and A a closed subalgebra of $C(X)$ containing the constants and separating the points of X. Let M denote the maximal ideal space of A. Then X is naturally embedded in M, and each f in A admits a natural continuous extension to M which we again write f. The "strong maximum principle for A" would read: "If x_0 lies in M but not in X, and if f is in A and is not constant, then $|f(x_0)| < \max|f(x)|$ taken over X." It does not hold for arbitrary A. The author defines

the "essential set" of A in X as the (unique) minimal closed subset E of X such that any f in $C(X)$ whose restriction to E lies in the restriction of A to E, must belong to A. He assumes: (1) the essential set of A is all of X; (2) A is maximal, i.e., contained in no proper closed subalgebra of $C(X)$ other than itself. Theorem: If A satisfies (1) and (2), then the strong maximum principle holds for A. Certain other results concerning essential sets are also given. *J. Wermer* (Providence, R.I.)

5909:

Domar, Yngve. Closed primary ideals in a class of Banach algebras. Math. Scand. **7** (1959), 109–125.

The author studies closed ideals in the Banach algebras F_p introduced by Beurling in Neuvième Congrès Math. Scand. Helsingfors, 1938, pp. 345–366 [Helsinki, 1939]. Let p be a measurable function on the line, bounded on each finite interval, such that (1) $p(x) \geqq 1$, (2) $p(x_1+x_2) \leqq p(x_1)p(x_2)$, and (3) $\int_{-\infty}^{\infty} [\log p(x)/(1+x^2)]\,dx < \infty$. F_p consists of all measurable functions f, with $\int_{-\infty}^{\infty} |f(x)|p(x)\,dx$ as norm. F_p is then an algebra under convolution. The author showed in Acta Math. **96** (1956), 1–66 [MR **17**, 1228], that every proper closed ideal of F_p is included in at least one regular maximal ideal. An ideal is 'primary' in F_p if it is contained in exactly one regular maximal ideal. For $p(x)\equiv 1$, it is well-known that each closed primary ideal is maximal. Put $\hat f(t)=\int_{-\infty}^{\infty} e^{-itx}f(x)\,dx$. Define for $n=0, 1, 2, \cdots$

$$I_n(t) = \{f\in F_p | \hat f(t) = \hat f'(t) = \cdots = \hat f^{(n)}(t) = 0\},$$
$$I_\infty(t) = \bigcap_n I_n(t).$$

Here $g^{(n)}$ means "nth derivative". For p growing fast enough as $|x|\to\infty$, the $I_n(t)$ are all defined and are closed primary ideals. The author now imposes sets (A) and (B) of additional conditions on p, which are rather involved. However, they are in particular fulfilled by the important examples: $\exp(|x|^\alpha), \exp\{|x|/\log|x|^\beta\}, 0<\alpha<1, \beta>1, |x|>C$. (A) asks among other things that $p^{(n)}$ exist for each n if $|x|\geqq x_n$ and $|p^{(n)}(x)|\geqq 1$ if $|x|\geqq x_n$. (B) asks that p be even and $\log p(x)$ be a convex function of $\log|x|$ for large $|x|$, and $[d\log p(x)/dx]/[x^{-1}\log|x|]\to+\infty$ as $|x|\to\infty$. Theorem 1: Let p satisfy (A). If I is a closed primary ideal included in $I_0(t)$ but not in $I_\infty(t)$, then $I=I_n(t)$ for some n. Theorem 2: If p satisfies (B) and if I is a closed primary ideal with I contained in $I_\infty(t)$, then $I=I_\infty(t)$.

Several applications to convolution equations $Q*f=0$, and to approximation problems are given.
J. Wermer (Providence, R.I.)

5910:

Tomiyama, Jun. Tensor products of commutative Banach algebras. Tôhoku Math. J. (2) **12** (1960), 147–154.

A. Hausner [Pacific J. Math. **7** (1957), 1603–1610; MR **20** #1931] and G. P. Johnson [Trans. Amer. Math. Soc. **92** (1959), 411–429; MR **21** #5910] have considered the B-algebra B of all Bochner integrable functions on a locally compact abelian group G with values in a complex commutative Banach algebra A (multiplication is convolution). They showed that $\hat G\times\mathfrak{M}$ is the space of regular maximal ideals of B, where $\hat G$ is the dual group of G and $\mathfrak{M}$ is the space of regular maximal ideals of A. Hausner [Proc. Amer. Math. Soc. **8** (1957), 246–249; MR **18**, 812] has shown that the B-algebra $C(\Omega, A)$ of all A-valued continuous functions on the compact space Ω has carrier space $\Omega\times\mathfrak{M}$. Both results are special cases of the following result of the author. Let A and B be commutative B-algebras, and suppose that the tensor product $T=A\otimes_\alpha B$ of A and B, for cross norm α not less than the λ-norm, is a B-algebra. Then its carrier space is $\mathfrak{M}(A)\times\mathfrak{M}(B)$. The author also shows T is regular if and only if A and B are. Conditions are given which make T semi-simple when A and B are semi-simple. *B. Yood* (Eugene, Ore.)

5911:

Takenouchi, Osamu. Sur les algèbres de Hilbert. C. R. Acad. Sci. Paris **250** (1960), 3436–3437.

A Hilbert subalgebra $\mathscr{B}$ of a Hilbert algebra $\mathscr{A}$ is defined to be a *-subalgebra of $\mathscr{A}$ endowed with the pre-Hilbert space structure induced by $\mathscr{A}$. If $\mathfrak{H}$ is the completion of $\mathscr{A}$, let the completion of such a $\mathscr{B}$ in $\mathfrak{H}$ be $\mathfrak{H}_\mathscr{B}$, and let $E_\mathscr{B}$ be the projection of $\mathfrak{H}$ onto $\mathfrak{H}_\mathscr{B}$. The author asserts, as his main result, that every $\mathscr{B}$-bounded element is $\mathscr{A}$-bounded and that $E_\mathscr{B}$ maps $\mathscr{A}$-bounded elements onto $\mathscr{B}$-bounded elements. This generalizes a result of Dixmier [Bull. Soc. Math. France **81** (1953), 9–39; MR **15**, 539] and Umegaki [Tôhoku Math. J. (2) **6** (1954), 177–181; MR **16**, 936]. Statements of three theorems follow concerning algebras generated by operators of left and right multiplication.
D. A. Edwards (Newcastle upon Tyne)

5912:

Misonou, Yoshinao. On the direct product of factors. Sûgaku **8** (1956/57), 32–33. (Japanese)

In the earlier paper [Tôhoku Math. J. (2) **6** (1954), 189–204; MR **16**, 1125] the author has shown general properties of the direct product of von Neumann algebras. This paper concerns some special properties of the direct product of finite factors. The same results have been published in English by the same author [ibid. **8** (1956), 63–69; MR **18**, 494]. *Z. Takeda* (Hitachi)

5913:

Ono, Tamio. Local theory of rings of operators. II. J. Math. Soc. Japan **10** (1958), 438–458.

There are two parts of this paper. In part one it is shown that a semi-finite AW^*-algebra satisfying a condition of Feldman [Duke Math. J. **23** (1956), 303–307; MR **17**, 1229] is a W^* algebra. In part two, the method of "local rings" is applied to the problem of spatial isomorphism of AW^*-algebras acting in Hilbert spaces. The theory of the coupling operator is generalized, yielding results similar to those of Pallu de la Barrière [Bull. Soc. Math. France **82** (1954), 1–52; MR **16**, 491], the reviewer [Trans. Amer. Math. Soc. **75** (1953), 471–504; **79** (1955), 389–400; MR **15**, 539; **17**, 66], and R. Kadison [Canad. J. Math. **7** (1955), 322–327; MR **17**, 178]. A feature of the author's method is the proof of dimension inequalities without use of the theory of unbounded operators.

Some related results may be found in R. Kadison, Ann. of Math. (2) **66** (1957), 304–379 [MR **19**, 665]. The theory of "local rings" may be found in the first paper of this series in same J. **10** (1958), 184–217 [MR **21** #5916].
E. L. Griffin, Jr. (Ann Arbor, Mich.)

5914:

Takesaki, Masamichi. On the Hahn-Banach type theorem and the Jordan decomposition of module linear mapping over some operator algebras. Kōdai Math. Sem. Rep. **12** (1960), 1–10.

Let A be a commutative AW^*-algebra, E a normed two-sided A-module and V a two-sided A-submodule of E. Every bounded A-linear and A-valued mapping on V can be extended to a bounded A-linear and A-valued mapping on E with preservation of the norm. Let **M** be a C^*-algebra with unit containing **A** as a C^*-subalgebra. If θ is a bounded A-linear and A-valued *-mapping on **M**, then there exist two positive elements a_1 and a_2 in **A** and two expectations θ_1 and θ_2 from **M** to **A** such that $\theta=a_1\theta_1-a_2\theta_2$, an expectation or generalized state from **M** to **A** being a bounded positive A-linear and A-valued mapping θ on **M** such that $\theta(a)=a$ for every $a\in$ **A**.

L. Nachbin (Waltham, Mass.)

5915:

Nicolescu, Lilly Jeanne. On some properties of the direct second order differentials in Gâteaux's sense. Rev. Math. Pures Appl. **4** (1959), 271–276.

A continuation of the author's earlier work in same Rev. **3** (1958), 217–223 [MR **21** #5919]. Three theorems are proved concerning functions from the complex numbers to a complex Banach space, or from one Banach space to another, relating types of differentiability: direct second order, Gateaux, weak, strong, uniform in compact sets.

E. R. Lorch (New York)

CALCULUS OF VARIATIONS

5916:

Fleming, W. H. Irreducible generalized surfaces. Riv. Mat. Univ. Parma **8** (1957), 251–281.

Until recently, an existence theory for problems of the calculus of variations which concern parametric surfaces of bounded topological type was known only in the classical case of the least area problem. One object of the program initiated 25 years ago with the reviewer's generalized curves was to provide an automatic existence theory for all such problems, even without assuming regularity. The author contributes in no small measure to this program, by completing in a number of respects the theory presented in the reviewer's memoir Mem. Amer. Math. Soc. No. 17 (1955) [MR **19**, 559]. For regular problems, the only alternative approach is that recently developed by Sigalov [Uspehi Mat. Nauk **12** (1957), no. 1 (73), 53–98; MR **19**, 560]; it contains an error not yet satisfactorily corrected [ibid. **15** (1960), no. 1 (91), 261; MR **22** #4000].

In the present paper, the results are, on the whole, considerably sharper than in the reviewer's memoir. Boundary identifications are now effected by piecewise linear substitutions, and the rims need be neither simple, nor two-dimensionally thin; moreover, by means of the concept of a limit boundary, the setting is generalized even further. Finally, the paper eliminates the need for a supplementary note by which the reviewer originally intended to justify a lemma stated without proof in establishing one of the final theorems.

On the other hand, the main theorem of the paper concerns only irreducible generalized surfaces (if the rims are devoid of multiple points, a slightly weaker assumption is made). Therefore, for an arbitrary generalized surface L with given rims and topological type, the author deduces only that it has the form L_1+L_2, where L_2 is a generalized surface, which has the given rims and topological type, and possesses a satisfactory micro-representation; whereas the reviewer's memoir treated such questions as whether L_1 can be chosen as a generalized sphere.

In regard to methods, apart from machinery already used by the reviewer, the author exploits conformal mapping for polyhedra of higher topological types (in a way similar to his recent extension [Trans. Amer. Math. Soc. **90** (1959), 323–335; MR **21** #2724] of Morrey's theorem), and also a pinching process which is well-known in other connections.

L. C. Young (Bloomington, Ind.)

5917:

De Vito, Luciano. Alcune osservazioni su un problema di Mayer per gli integrali multipli. Rend. Mat. e Appl. (5) **18** (1959), 313–350.

This paper remarks on possible weakening of the hypotheses in a previous paper by the author in Ricerche Mat. **8** (1959), 3–23 [MR **21** #7461]. The first change is that the assumption of uniqueness of the solution of the differential equation $u'=f(x,z,p,u)$ may be omitted, and another condition on f may be weakened. Also, the problem is generalized so as to include problems of minimizing r-tuple integrals over a domain of arbitrary shape (but sufficiently regular). For the Mayer problem, the domain of the variables x is still restricted to be rectangular. The final step is to use transformations of the domain. The last 20 pages of the note are devoted to a discussion of transformations of multiply connected plane regions into a simply connected region. The type of transformations admitted here was not clear to the reviewer.

L. M. Graves (Chicago, Ill.)

5918:

Picone, Mauro. Su un criterio sufficiente in un classico problema di calcolo delle variazioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 131–138.

A sufficient condition is given that a function $z_0(x,y)$ shall minimize a double integral $I=\iint f(x,y,z,z_x,z_y)\,dx\,dy$ with respect to fixed boundary values. This condition states that there exists an invariant integral I^* with integrand h of the first degree in the partial derivatives z_x and z_y, such that $I^*(z_0)=I(z_0)$, and $f(x,y,z,u,v)\geq h(x,y,z,u,v)$ for all (x,y,z,u,v). The proof is obvious, and does not use the notion of a field, nor restrict the minimizing surface $z_0(x,y)$ to lie interior to the domain of admissible points. The author states that the idea is applicable to cases where there are r independent variables $(r\geq 2)$ and q dependent variables $(q>1)$, but does not mention that when $s>1$ (where s is the smaller of q and r) the integrand of an invariant integral may advantageously contain terms up to degree s in the partial derivatives of z. Nor is there mention of the additional limitations on the choice of the coefficients in these cases. For these matters, reference should be made to the work of A. W. Landers, *Contributions to the calculus of variations, 1938–1941*, pp. 175–207 [Univ. Chicago Press, Chicago, Ill., 1942;

MR **4**, 46]. See also: M. R. Hestenes and E. J. McShane, Trans. Amer. Math. Soc. **47** (1940), 501–512 [MR **2**, 119] and G. A. Bliss, Amer. Math. Monthly **49** (1942), 77–89 [MR **3**, 248]. *L. M. Graves* (Chicago, Ill.)

5919:

Morrey, Charles B., Jr. Multiple integral problems in the calculus of variations and related topics. Ann. Scuola Norm. Sup. Pisa (3) **14** (1960), 1–61.

In the present paper the author gives a simplified account, with full proofs, of his results concerning non-parametric multiple integrals, which have appeared separately elsewhere in more detail [Comm. Pure Appl. Math. **9** (1956), 499–508; MR **19**, 408; and earlier papers]. This excellent presentation was prompted by a series of lectures on the subject given by the author in Pisa in 1958.

In chapter I the well-known class B_λ, $\lambda \geqq 1$, of real functions $z(x)$ $(x=(x_1, \cdots, x_\nu) \in G$, G a bounded domain in $E_\nu)$ is introduced, and the properties of these functions are investigated in detail. In particular, beside the properties of approximation of these functions by means of the h-averaged functions z_h, the author discusses their property of forming a normed linear Banach space B_λ (a Hilbert space for $\lambda=2$).

In chapter II multiple integrals (*) $I[z]=(G)\int f(x, z(x), \nabla(x))\, dx$ are studied, with $dx=dx_1 \cdots dx_\nu$, $z(x) \in B_\lambda$, $\nabla(x)$ the vector of the generalized first partial derivatives of $z(x)$, f continuous in (x, z, p) and convex in p for every (x, z), $x \in G$. Also, analogous integrals (*) are studied where $z=(z_1, \cdots, z_N)$, $z_j \in B_\lambda$, and $\nabla(x)$ is the matrix of the first partial derivatives. Under light assumptions, integrals (*) are proved to be lower semicontinuous with respect to weak convergence in B_λ. General existence theorems of the absolute minimum of (*) in closed classes F of functions $z(x) \in B_\lambda$ are then deduced. The proofs are elegant. One assumption is that $f(x, z, p) \geqq f_0(p)$, where $f_0(p)$ is convex and $f_0(p)/p \to +\infty$ as $p \to \infty$. The requirements for the classes F are of general nature, essentially implying that the B_λ norms of the functions z in F are uniformly bounded. Beside the usual situations, cases of mixed boundary conditions (boundary partly free and partly fixed) are included; in particular, for $\nu=2$, cases where the boundary is not a continuous curve.

In chapter III conditions more general than convexity are studied in relation to lower semicontinuity (convexity in all the components of p is not a necessary condition for $N>1$, as is well known in the parametric case). A necessary and sufficient condition is given for $N>1$ and weak convergence. Also, concepts of quasi-convexity and weak quasi-convexity in relation to a modified form of weak convergence are brought to bear on lower semi-continuity.

In chapter IV the differentiability of the solutions of variational problems is studied for $\nu=2$, $N=1$ (surfaces in non-parametric form) as a consequence of Morrey's inequality for the Dirichlet integral (Dirichlet integral growth theorem). Essentially under the same assumptions above, a function $z(x)$ for which I is stationary is proved to have first partial derivatives satisfying a Hölder condition. This typical result is only one among a great many others due to the author, De Giorgi, J. Nash, and others on the subject [a more stringent result, announced by the author in summer 1960, is to appear].

In chapter V the author applies the same variational technique to the theory of harmonic integrals over a compact manifold M of class C_k, $k \geqq 1$. In particular, an existence theorem is proved for the minimum of the Dirichlet integral for differential r-forms in M, and a complement to the orthogonal decomposition theorem of K. Kodaira for r-forms in M.

L. Cesari (Ann Arbor, Mich.)

GEOMETRY

See also 5633, 5933.

5920:

Myller, A. Triangles en géométrie aréolaire. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **10** (1959), 269–276. (Romanian. Russian and French summaries)

5921:

Frost, Percival. ★An elementary treatise on curve tracing. Revised by R. J. T. Bell. 5th ed. Chelsea Publishing Co., New York, 1960. xvi+210 pp. $3.50.

A reproduction of the 5th edition [Macmillan, London, 1917]; the first edition was dated 1892.

5922:

Bartoš, Pavel. Linear sets of directly similar figures in the plane. Časopis Pěst. Mat. **85** (1960), 198–199. (Slovak)

Let L be a linear system of figures U in the complex z-plane so that (1) any two figures in L are directly similar, and (2) if in the three figures U_i $(i=1, 2, 3)$ there are three different points A_i corresponding to each other under similarity, then the A_i are collinear. The following two cases are considered. (I) Two similarities $z'=a_jz+b_j$ $(j=1, 2)$ between systems in L have a common fixed point. (II) There is no general fixed point common to all similarities; there are however two different fixed points: (a) one of them ∞; (b) both finite. Conditions on the coefficients of the similarities are established.

H. Schwerdtfeger (Montreal)

5923:

Herrmann, Manfred. Eine Bemerkung über Plückersche Äquivalenzzahlen. Math. Ann. **139**, 180–183 (1960).

The problem here is to find expressions for the numbers of nodes, cusps, bitangents and inflexions "equivalent" (for the purposes of the Plücker formulae, etc.) to any given singularities of a plane curve. It is elementary that the total "equivalent" number of double points (including cusps) is given by $d=\sum \binom{i}{2}$, where i is the multiplicity of a point and the summation is over all singular points, "neighbour" as well as ordinary; and Keller [same Ann. **118** (1943), 626–628; MR **6**, 101] showed that the "equivalent" number of cusps is $e=\sum' i$, the summation being over all satellite points.

Dually, the author defines d', the total "equivalent" number of double (including inflexional) tangents, and e', the "equivalent" number of inflexions. He shows that e' is the sum of the multiplicities of all "neighbour" points

which are either satellite or lie on a tangent, but not (in either case) proximate to an ordinary point. For d' he has the following result:

If a tangent t_λ has one or more points of contact $S_{\lambda\mu}$ ($\mu=1, \cdots$) and in $S_{\lambda\mu}$ touches one or more branches $Z_{\lambda\mu\nu}$ ($\nu=1, \cdots$), the contribution of t_λ to d' is

$$d'_\lambda = \binom{\sum_\mu \sum_\nu l_{\lambda\mu\nu}}{2} - \sum_\mu \binom{\sum_\nu k_{\lambda\mu\nu}}{2} + \sum_\mu d_{\lambda\mu},$$

where $k_{\lambda\mu\nu}$, $l_{\lambda\mu\nu}$ are the order and class of the branch $Z_{\lambda\mu\nu}$, and $d_{\lambda\mu}$ is the contribution of $S_{\lambda\mu}$ (with all its neighbourhoods) to d. *P. Du Val* (London)

5924:

Mammana, Carmelo. Su certe ipersuperficie, analoghe alle jacobiane, legate ad una corrispondenza cremoniana. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **27** (1959), 339–346.

Es seien $x_i'=f_i(x_0, \cdots, x_r)$; $x_i=g_i(x_0', \cdots, x_r')$ ($i=1, \cdots, r$) die Gleichungen einer Cremona-Transformation zwischen zwei r-dimensionalen Räumen und ihrer Umkehrung. Die f_i seien vom n-ten, die g_i vom m-ten Grade. Es ist

$$f_i(g_0, \cdots, g_r) = H(x_0', \cdots, x_r')x_i',$$
$$g_i(f_0, \cdots, f_r) = K(x_0, \cdots, x_r)x_i.$$

Verf. setzt sich die Aufgabe diese Polynome H und K zu untersuchen. Er zeigt, dass $K(x_0, \cdots, x_r)$, $H(f_0, \cdots, f_r)$ und die Jacobische Form der Transformation dieselben irreduziblen Komponenten haben, nur mit anderen Exponenten. Nur wenn $n=m=r$ ist, ist $K(x_0, \cdots, x_r)$ bis auf einen Faktor gleich der Jacobischen Form.

Es sei $K(x)=\prod K_i^{\nu_i}(x)$ und $H(x')=\prod H_i^{\nu_i'}(x')$, wo die K_i und H_i irreduzible Formen jeweils des Grades μ_i und μ_i' sein. Dann sind die grössten gemeinsamen Teiler der ν_i und der ν_i', ebenso der grösste gemeinsame Teiler der μ_i und der μ_i', jeweils einander gleich. Ferner sind die Anzahlen einander gleich.

Es sei $H_i(f)=a_i \prod K_j^{p_{ji}}(x)$ und $K_j(g)=b_j \prod H_i^{q_{ij}}(x')$. Die Matrizen $P=((p_{ji}))$ und $Q=((q_{ij}))$ sind quadratisch. Ihre Determinanten sind $\pm n$ und $\pm m$.

Verf. gelangt zu allen diesen Ergebnissen durch einfache algebraische Methoden. *O. H. Keller* (Halle)

5925:

Andreatta, Antonio. Semplici considerazioni topologiche utili nella geometria sulla curva algebrica. Boll. Un. Mat. Ital. **15** (1960), 286–290. (English summary)

Author's summary: "Tra due superficie topologiche serrate e chiuse si considerano certe corrispondenze $(n, 1)$ dotate di un numero finito di punti di diramazione. Si stabilisce la relazione che intercede tra gli ordini di connessione delle due superficie, traendone classiche proposizioni di geometria sulla curva algebrica."

5926:

Salmon, George. ★A treatise on the higher plane curves: intended as a sequel to "A treatise on conic sections". 3rd ed. Chelsea Publishing Co., New York, 1960. xix+395 pp. $4.95.

A reproduction of the third edition [Hodges, Foster and Figgis, Dublin, 1879] of this work. List of chapters: (I) Coordinates; (II) General properties of algebraic curves; (III) Envelopes; (IV) Metrical properties; (V) Cubics; (VI) Quartics; (VII) Transcendental curves; (VIII) Transformation of curves; (IX) General theory of curves.

5927:

Cremona, Luigi. ★Elements of projective geometry. 3rd ed. Translated by Charles Leudesdorf. Dover Publications, Inc., New York, 1960. xx+302 pp. $1.75.

Unaltered republication of the 3rd English edition [Oxford Univ. Press, Oxford, 1913]. Contents (in brief): homology; geometric forms; duality; projective forms; harmonic forms; anharmonic ratios; involution; theorems of Pascal, Brianchon, Desargues; self-corresponding elements; pole and polar; centre and diameters of a conic; polar reciprocity; foci.

5928:

Horadam, A. F. Clifford groups in the plane. Quart. J. Math. Oxford Ser. (2) **10** (1959), 294–295.

The groups in question are the Hessian group of order 216, and some related groups. *J. A. Todd* (Cambridge, England)

5929:

Court, Nathan Altshiller. Sur la cubique gauche, application de la transformation harmonique. Mathesis **68** (1959), 110–127.

Author's summary: "La cubique gauche C_3 peut se définir comme le lieu du pôle harmonique d'un plan variable passant par une droite fixe s par rapport à un tétraèdre donné (T) inscrit dans C_3 (§§ 2, 3). En étudiant la cubique à ce point de vue, on obtient bon nombre de résultats dont voici quelques exemples, choisis un peu au hasard.

"(i) Les droites s', s'', s''' qui correspondent à la droite s dans les trois homologies harmoniques gauches ayant pour axes les trois couples d'arêtes opposées de (T) sont trois sécantes de la cubique gauche (§ 3c).

"(ii) Les quatre droites s, s', s'', s''' et les quatre droites s_a, s_b, s_c, s_d qui correspondent à s dans les quatre homologies ayant pour centres et plans d'homologie les sommets et les faces de (T) respectivement opposées, appartiennent aux deux systèmes complémentaires d'une même quadrique réglée.

"Les douze points d'intersection des droites s', s'', s''' avec les droites s_a, s_b, s_c, s_d sont les pôles des arêtes de (T) par rapport aux quatre coniques suivant lesquelles la cubique est projetée des sommets de (T) sur les faces opposées (§ 4d).

"(iii) Si les constantes des quatre homologies centrales considérées dans (ii) sont égales à $-1/3$, les quatre droites que ces homologies font correspondre à la droite s sont quatre sécantes de la cubique (§ 5b).

"(iv) Étant donnés deux tétraèdres de Moebius, on peut circonscrire à chacun d'eux une cubique gauche, et une seule, de telle manière que les quatre droites joignant les couples de sommets correspondants de ces solides touchent les deux courbes aux sommets respectifs (§ 8b)."

5930:

Guber, Siegfried. Strecken- und Winkelübertragung mit Lineal und Eichmass in der absoluten Geometrie. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1959**, 251–261 (1960).

H. G. Forder [Math. Gaz. **23** (1939), 465–467] has shown that in Bolyai's absolute geometry, a fixed segment (or a gauge) and a ruler are sufficient to construct: (1) on any ray, a segment congruent to a given segment; and (2) an angle, with a given vertex and side, congruent to a given angle. The author obtains these results by a different method. *M. W. Al-Dhahir* (Toronto)

5931:

Wilniewczyc, R. Geometry of a group of affine transformations for which a parabola remains invariant. Prace Mat. **2** (1958), 287–298. (Polish. Russian and English summaries)

The author develops the geometry of the interior of the parabola $x^2-2y=0$ regarded as Klein space of the two-parameter transformation group $\bar{x}=\lambda x+k$, $\bar{y}=k\lambda x+\lambda^2y+\frac{1}{2}k^2$ $(\lambda>0)$. The invariants $s=\int(2y-x^2)^{-2}\,dx$ and $K=(x-y')/\sqrt{(2x-x^2)}$ (called arc and curvature) of a curve $y=y(x)$ are determined. $K, dK/ds, \cdots, d^nK/ds^n, \cdots$ form a complete system of invariants. The curves of constant curvature are investigated. An invariant area with element $d\sigma=-dxdy/(2y-x^2)^{3/2}$ is found. *A. Goetz* (Wroclaw)

5932:

Greenberg, Leon. Discrete groups of motions. Canad. J. Math. **12** (1960), 415–426.

Discrete groups F of rigid motions of the hyperbolic plane are called hyperbolic groups. Their elements are of the form $S(z)=(az+\bar{b})/(bz+\bar{a})$ or $T(z)=(c\bar{z}+\bar{d})/(d\bar{z}+\bar{c})$, with $a\bar{a}-b\bar{b}=c\bar{c}-d\bar{d}=1$. If F is finitely generated, it is known [Nielsen, Den 11te Skandinaviske Matematikerkongress (Trondheim, 1949), pp. 61–70, Johan Grundt Tanums Forlag, Oslo, 1952; MR **15**, 21] that one may select the generators as a_i, b_i $(i=1, 2, \cdots, p)$ with commutators $k_i=a_ib_ia_i^{-1}b_i^{-1}$, S_j $(j=1, 2, \cdots, d)$ of finite orders n_j and c_ν $(\nu=1, 2, \cdots, r)$ in such a way that

$$k_1\cdots k_pS_1\cdots S_dc_1\cdots c_r=S_j^{n_j}=1$$

are the defining relations. Such a group is denoted by $F(p; n_1, \cdots, n_d; r)$. In particular, the modular group is of type $F(0; 2, 3; 1)$, being generated by $S_1=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$,

$S_2=\begin{pmatrix}-1&1\\-1&0\end{pmatrix}$ and $c=\begin{pmatrix}-1&0\\-1&-1\end{pmatrix}$, with $S_1^2=S_2^3=S_1S_2c=1$.

The set of limit points of such a group G is denoted by L_G. If L_G consists of at most two points, G is called quasi-abelian. Using results of Nielsen [Mat. Tidskr. B **1948**, 49–56; MR **10**, 590], Bundgaard and Nielsen [ibid. **1951**, 56–58; MR **14**, 15], Coxeter [J. Math. Pures Appl. (9) **37** (1958), 317–319; MR **20** #5238] and K. Goldberg [J. Washington Acad. Sci. **46** (1956), 337–338; MR **19**, 123], the author proves results like the following: If $F(p; n_1, \cdots, n_d; r)$ is hyperbolic, then the centralizer of any element is cyclic; the possible finite orders are the divisors of $n_1, \cdots, n_d$. Any finite subgroup is cyclic, being conjugate to a subgroup of some $\langle S_j\rangle$. If S and T are finitely generated subgroups of a discrete group, then $S\cap T$ is finitely generated. If H is a finitely generated subgroup of G and $L_G=L_H$, then $[G:H]$ is finite. If H and K are finitely generated non-quasi-abelian subgroups of a discrete group, then $L_H=L_K$ is equivalent to the existence of a group J, normal and of finite index in both H and K. If H is a finitely generated subgroup of a finitely generated non-quasi-abelian group G, then the index $[G:H]$ is finite if and only if H is contained in no infinitely generated subgroup of G. If H is a finitely generated subgroup of a non-quasi-abelian group G and has a non-trivial intersection with every non-cyclic subgroup of G, then $[G:H]$ is finite. *E. Grosswald* (Philadelphia, Pa.)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 5593, 5879.

5933:

Kárteszi, F. Über ein elementargeometrisches Problem. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **2** (1959), 49–60.

The author defines a rectangle of support of a plane convex polygon as a rectangle all of whose sides are lines of support of the polygon. If at least one of the sides of the rectangle contains a side of the polygon, the former is called an osculating rectangle of support. From the assumption that all rectangles of support of an n-gon are squares a classification of such polygons is obtained for $n=3, 4, 5, 6$. It depends on the number of osculating squares and their relative positions. The results proved are that, under the given restriction, for $n=3$ there exists no triangle, for $n=4$ only the square is permissible, for $n=5$ there is a unique representation of the possible pentagons as the common part of two osculating squares in a special position, while for $n=6$ four different classes are found.

A final remark indicates how some of the theorems used can be generalized if an analogously defined regular n-gon of support takes the place of the square of support. *C. J. Scriba* (Toronto)

5934:

Goldberg, Michael. Rotors in polygons and polyhedra. Math. Comput. **14** (1960), 229–239.

The author summarizes 36 papers (including 8 of his own) on curves of constant width and other rotors. For instance [M. Goldberg, Amer. Math. Monthly **64** (1957), 71–78; MR **18**, 668] there is a figure formed by $n\pm1$ equal circular arcs, which can be rigidly moved so that it has all the sides of a fixed regular n-gon as lines of support. For each such figure in the plane, there is an analogous figure on the surface of a sphere (having all the sides of a spherical polygon as "great circles of support"); but now the arcs are not plane, and therefore not circular. There are also solid rotors for the regular tetrahedron, octahedron, and cube, but not for the remaining regular solids.

If a rotor in a regular polygon is held fixed while the polygon is rotated about it, all the vertices of the polygon trace the same curve. Therefore, if this curve is fixed, the polygon can be rotated within it while all the vertices lie on the curve. This idea has useful mechanical applications. *H. S. M. Coxeter* (Toronto)

5935:

Nieminen, Toivo. On decompositions of simplexes and convex polyhedra. Soc. Sci. Fenn. Comment. Phys.-Math. **20** (1957), no. 5, 24 pp.

The first three sections contain standard material on convex polyhedra and their decompositions. The fourth section contains an apparently new result on decompositions which was motivated by a treatment of integration due to R. Nevanlinna: Let S denote the m-dimensional simplex $\{x=(x^1, \cdots, x^m)\in E^m : x^i \geqq 0, \sum x^i \leqq 1\}$, K the parallelepiped $\{x: 0 \leqq x^i \leqq 1\}$ and (for $j=1, \cdots, m$) P_j the intersection of K with the "strip" $\{x: j-1 \leqq \sum x^i \leqq j\}$. Then there is a sequence $D_1, D_2, \cdots$ of cellular decompositions of S such that each polyhedron in D_n is obtained from some P_j by a translation and homothety in the ratio $\pm 1/n$, and such that always D_{2n} is a subdecomposition of D_n. *Victor Klee* (Seattle, Wash.)

DIFFERENTIAL GEOMETRY

See also 5550, 5926, 5931, 5979.

5936:

Eisenhart, Luther Pfahler. ★A treatise on the differential geometry of curves and surfaces. Dover Publications, Inc., New York, 1960. xiv+474 pp. $2.75.

Unaltered republication of 1909 edition [Ginn and Co., Boston]. Contents (in brief): Curves in space; linear element of a surface, conformal representation; local geometry of a surface; equations of Gauss and Codazzi, moving trihedral; systems of curves, geodesics; quadrics, ruled surfaces, minimal surfaces; constant curvature, W-surfaces, plane or spherical lines of curvature; deformation of surfaces; rectilinear congruences; cyclic systems; triply orthogonal systems of surfaces.

5937:

Mihăilescu, Tiberiu. Formes différentielles quadratiques et courbures en géométrie projective. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 165–168.

The method of the moving frame (repère mobile) is applied to an investigation of the properties of projective differential invariants of a surface S imbedded in a 3-dimensional euclidean space. Families of such frames depending on p parameters ($p \leqq 12$) are attached to each point of S. A number of exterior differential forms are obtained, giving rise to a finite projective invariant K, called the projective curvature of S. The properties of K are studied in some detail. *H. Rund* (Durban)

5938:

Godeaux, Lucien. Sur les surfaces représentant les tangentes asymptotiques de la surface de Steiner. Atti Accad. Ligure **15** (1959), 104–109. (Italian summary)

The surfaces which represent the tangents to the asymptotics of the surface of Steiner on the Klein hyperquadric Q are rational of order 15. Let u, v describe these asymptotics, let U and V describe the points on Q which represent the tangents to u and v. Let $U_1, U_2, \cdots$ describe the successive Laplace transforms of U in the sense of v and $V_1, V_2, \cdots$ those of V in the sense of u. There is a biaxial, harmonic homography, which transforms Q into itself, in which the surfaces (U) and (V), (U_1) and (V_1), (U_2) and (V_2), $\cdots$, correspond. *A. Schwartz* (New York)

5939:

Karapetyan, S. E. The transformation of congruences with the help of half-quadrics. Mat. Sb. (N.S.) **50 (92)** (1960), 109–116. (Russian)

In the paper Dokl. Akad. Nauk SSSR **117** (1957), 177–179 [MR **20** #302] were introduced half-quadrics (reguli) of Lie of ruled surfaces ($\omega_2^4=\lambda\omega_1^3$) of a line congruence, given in a system with coordinate tetrahedron $\{A_i\}$ satisfying $dA_i=\omega_i^k A_k$ ($i, k=1, 2, 3, 4$). In the present paper such generators of a half-quadric are determined which are at the same time also the characteristics of this half-quadric for a motion of the latter in a given direction, here given by $\omega_2^4=\mu\omega_1^3$. This leads to a certain relationship between λ and μ. To every ray of the congruence for given λ is thus associated another ray, the characteristic, which describes a second congruence, called the transform of the first congruence with respect to the half-quadrics. These transformed rays and surfaces are especially investigated as to what are called asymptotic ruled surfaces, which are the surfaces which correspond to the asymptotic lines of the first focal surface of the congruence. *D. J. Struik* (Cambridge, Mass.)

5940:

Dražila, Pavel [Dragila, Pavel]. On congruences whose focal surfaces overlap. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 3, 15–17. (Russian)

L'auteur poursuit sa méthode à généraliser les solutions des problèmes projectifs en théorie des surfaces et congruences. Le premier l'a fait Bianchi, quand il a élargit les transformations des surfaces à courbure constante négative en transformations des surfaces à courbure constante positive. Il lui a fallut passer des équations de Moutard

$$\frac{\partial^2\vartheta}{\partial u\partial v} = M\vartheta$$

aux équations

$$\frac{\partial^2\vartheta}{\partial u^2}+\frac{\partial^2\vartheta}{\partial v^2}+\sinh\vartheta\cosh\vartheta = 0.$$

En géométrie projective ça se fait beaucoup plus simplement. Il suffit de réduire par le choix convenable des paramètres u, v les fonctions $U=\varphi(u)$, $V=\psi(u)$ à $U=-1$, $V=-1$ au lieu de $U=V=1$ et l'équation des lignes asymptotiques devient $du^2+dv^2=0$ au lieu de $du^2-dv^2=0$. *S. P. Finikov* (Moscow)

5941:

Moór, A. Über die aus g_{ik} bestimmte kovariante Ableitung. Acta Math. Acad. Sci. Hungar. **11** (1960), 175–186. (Russian summary, unbound insert)

This paper deals with the theory of the covariant derivative as a branch of the theory of geometric objects. It starts from the three conditions to be satisfied, namely, (i) the covariant derivative of a vector must be a tensor of the second order, (ii) it must depend on the vector, its partial derivative, and also upon an auxiliary object g_{ij}

and its partial derivatives, and (iii) the covariant derivative must be a continuous function of all the elements which enter into it and is to be restricted to systems for which $\det(g_{ij}) \neq 0$. The first theorem proved is that if symmetric coefficients $\Gamma_j{}^i{}_k$ satisfy the usual law of transformation known in tensor analysis, and if they are functions of $g_{ab} = g_{ba}$ and of their partial derivatives, then they must coincide with the three-index symbols of Christoffel. In the remainder of the paper the form of the most general covariant derivative satisfying the three conditions is given. In particular, the extent to which certain elements of the covariant derivative must occur explicitly is given. *E. T. Davies* (Southampton)

5942:

Sen, R. N.; Sen, Hrishikes. On the geometry at a point of a hypersurface of a Riemannian space. Bull. Calcutta Math. Soc. **50** (1958), 193–203.

The totality of all contravariant vectors through any point of a Riemannian space V_n forms a projective space S_{n-1}, components of the vectors being homogeneous point coordinates in S_{n-1}. Ruse [Proc. Roy. Soc. Edinburgh Sect. A **62** (1944), 64–73; MR **6**, 106] has shown how aspects of the differential geometry of V_n may be interpreted in terms of the geometry of objects in S_{n-1}. In particular, corresponding to the metric and Riemann tensors of V_n, he has associated the fundamental hyperquadric of S_{n-1} and a certain quadratic line complex (Riemann complex) respectively. The present paper extends this idea to interpret the differential geometry at a non-singular point of a hypersurface V_3 of a Riemann space V_4 in projective language. The projective geometry of S_3 (corresponding to V_4) induces a projective geometry in S_2 (corresponding to V_3). The fundamental conic in S_2 is obtained as the section by S_2 of the fundamental quadric in S_3. The tensor whose components are the coefficients of the second fundamental form of V_3 also defines a conic in S_2. Various polarity and conjugate relationships relative to these conics are derived. Geometric properties of the Riemann complexes in S_2 and in S_3 are obtained, utilizing the Gauss-Codazzi equations. *A. Fialkow* (Brooklyn, N.Y.)

5943:

Sen, Hrishikes. On the geometry at a point of a subspace of a Riemannian space. Proc. Nat. Inst. Sci. India. Part A **25** (1959), 356–363.

The results of the preceding paper [#5942 above] are generalized and extended to the case of a subspace V_n of a Riemannian space V_m. *A. Fialkow* (Brooklyn, N.Y.)

5944:

Levin, Yu. I. Affine connectivities adjoint to a skew-symmetric tensor. Dokl. Akad. Nauk SSSR **128** (1959), 668–671. (Russian)

The author studies the structure of spaces on which are given an affine connection $\Gamma_{ij}{}^k$ and a non-degenerate skew-symmetric tensor a_{ij} satisfying the condition

$$(1) \qquad \nabla_k a_{ij} = \mu T_{ijk} \quad (i, j, k = 1, \cdots, n),$$

where $T_{ijk} = \frac{1}{3}(\partial a_{ij}/\partial x_k + \partial a_{jk}/\partial x_i + \partial a_{ki}/\partial x_j)$ and μ is a real number. Denote such a space by $L_\mu{}^n$. Then the author considers conditions under which the tensor a_{ij} has, in properly chosen coordinates, a special form. For example (theorem 1), necessary and sufficient conditions are given that in suitable coordinates $a_{ij} = e^\sigma c_{ij}$, c_{ij} being constants. Or it is shown (theorem 4) that if a_{ij} has vanishing covariant derivative, $\nabla_k a_{ij}$, then the Ricci curvature R^k_{ijk} of the connection vanishes.

Considering the class of connections for which (1) is satisfied with a fixed a_{ij}, the final theorem (theorem 6) states that there exists a flat connection with the property that $a_{ij} = c_{ij}\phi_i\phi_j$, where the c_{ij} are constants and ϕ_j functions such that $\Gamma_{ij}{}^k = \delta_j{}^k(\partial \ln \phi_j/\partial x_i)$. The existence of such functions ϕ_j ($j = 1, \cdots, n$) is implied by a flat connection, i.e., one whose curvature tensor vanishes.

W. M. Boothby (St. Louis, Mo.)

5945:

Aeppli, Alfred. Einige Ähnlichkeits- und Symmetriesätze für differenzierbare Flächen im Raum. Comment. Math. Helv. **33** (1959), 174–195.

Let F_1, F_2 be two oriented hypersurfaces of class C^2 in R^{n+1} $(n \geq 2)$ given by $x_i{}^j = x_i{}^j(u^1, \cdots, u^n) = x_i{}^j(u)$ $(j = 1, \cdots, n+1; i = 1, 2)$. Let T be a topological map of F_1 on F_2 of the form $x_1{}^j(u) = f(u)x_2{}^j(u)$ $(j = 1, \cdots, n+1)$, briefly $x_1(u) = f(u)x_2(u)$, where $f(u)$ is of class C^2 and the topological map is regular, i.e., its Jacobian does not vanish. Assume moreover that the shadow boundary of F_i with O as light source contains no interior points. If $f(u) > 0$ and the mean curvatures $H_i(u)$ of F_i satisfy $H_1(u) = f(u)H_2(u)$, then T is a dilation (or $f(u)$ is constant) (1) if the F_i are closed or (2) if the F_i have boundaries and parallel normals at corresponding points of these boundaries. This also holds for $f(u) < 0$ and yields, applied to a single surface, that O is its center. (For closed F_i this was proved independently by Chin-Shui Hsü, Proc. Amer. Math. Soc. **10** (1959), 324–328 [MR **21** #6601].)

For a fixed closed F_1 consider the class $K(F_1, O)$ of all F_2 such that the above conditions are satisfied with $f > 0$. Then for all $F_2 \in K(F_1, O)$ the inequality

$$\int_{F_1} (H_1 - fH_2)(x \cdot n)\, dA \leq 0$$

holds, where n is the unit normal of F_1 and dA its area element, with equality only when f is constant. This theorem contains (1) and yields a characterisation of the unit sphere S with center O among all $F_2 \in K(S, O)$ as the surface maximizing $\int_S H_1\, dA$.

Theorem (1) is extended to the case of the νth elementary symmetric function of the principal curvatures under the following additional hypotheses: F_1, F_2 are similarly oriented convex surfaces and the family

$$x_t(u) = [(1-t) + tf]x_1(u) \quad (0 \leq t \leq 1)$$

consists of convex surfaces.

In addition there are several theorems which are implied by results of A. D. Alexandrov in his series of papers entitled "Uniqueness theorems for surfaces in the large", in particular theorems 6, 7 in part I [Vestnik Leningrad. Univ. **12** (1957), no. 19, 15–44; MR **19**, 167], which was evidently not available to the author, who submitted his paper in January, 1958.

H. Busemann (Los Angeles, Calif.)

5946:

Pogorelow, A. W. ★Einige Untersuchungen zur Riemannschen Geometrie im Grossen. Mathematische Forschungsberichte, VIII. VEB Deutscher Verlag der Wissenschaften, Berlin, 1960. 71 pp. DM 14.00.

Translation of *Nekotorye voprosy geometrii v celom rimanovom prostranstve* [Izdat. Har'kov. Univ., Kharkov, 1957; MR **20** #4303].

5947:

Rembs, Eduard. Über die Verbiegbarkeit der Rinnen. II. Math. Z. **73** (1960), 330–332.

A groove ("Rinne") is a three-times differentiable surface R containing a plane curve K such that the plane through K supports R along K and that K divides R into two regions of opposite non-vanishing Gauss curvature. Suppose K is closed and convex and its curvature is continuous and non-vanishing. Bend the plane region bounded by K into a non-singular region on a developable surface. The plane strip through K will be mapped onto a strip on the developable. The author proves that no surface through that strip can be isometric to R. [Cf. Part I, Math. Z. **71** (1959), 89–93; MR **21** #6009.]

P. Scherk (Toronto)

5948:

Wolf, Joseph A. Sur la classification des variétés riemanniennes homogènes à courbure constante. C. R. Acad. Sci. Paris **250** (1960), 3443–3445.

The author in this note studies certain classes of homogeneous Riemann manifolds. His main tool theorem may be stated after the following definition: An isometry of a Riemann manifold is called a Clifford translation if the distance between a point and its image is constant. The author then proves: Let $p: N \to M$ be a covering, let M be a homogeneous Riemann manifold and let the covering group be D. Then every element of D is a Clifford translation on N. From this the author obtains: (1) A homogeneous Riemann manifold of constant negative curvature is a hyperbolic space; (2) a homogeneous locally euclidean space is the product of a euclidean space and a euclidean torus; (3) a homogeneous Riemann manifold of constant positive curvature $k>0$ is determined up to isometry by its dimension and its fundamental group.

L. Auslander (New Haven, Conn.)

5949:

Huber, Alfred. Zum potentialtheoretischen Aspekt der Alexandrowschen Flächentheorie. Comment. Math. Helv. **34** (1960), 99–126.

If M is a (2-dimensional) manifold of bounded curvature in the sense of A. D. Alexandrov, then Rešetnyak [Dokl. Akad. Nauk SSSR **94** (1954), 631–634; MR **16**, 167] has shown that it is possible to introduce generalized local isothermal coordinates on M. In particular, in these coordinates, the distance $\rho(z_1, z_2)$ between points z_1, z_2 (on a neighborhood) can be represented in the form (1) $\rho(z_1, z_2) = \inf \int_\Gamma e^{u(z)}|dz|$, where $u(z)$ is the difference of two sub-harmonic functions and Γ varies over polygonal paths joining z_1, z_2. The present author's main result is a corresponding theorem in the large: An (open or closed) orientable manifold of bounded curvature is isometric to a space of the following type: on a Riemann surface R let there be defined a conformally invariant line-element $ds = e^{u(z)}|dz|$, where $u(z)$ is representable as the difference of two sub-harmonic functions of a local coordinate z, and let the distance $\rho(p, q)$ between points p, q of R be defined by $\rho(p, q) = \inf \int_\Gamma e^{u(z)}|dz|$, where L varies over piecewise analytic arcs joining p and q. Conversely, such a space is a manifold of bounded curvature. In addition to Rešetnyak's result, the proof of this theorem depends largely on properties of functions representable as the difference of subharmonic functions. The proof is finally reduced to an application of a result of Morrey characterizing analytic functions [Trans. Amer. Math. Soc. **43** (1938), 126–166]. There is also derived a potential-theoretic definition of the lengths of an arc in the metric (1).

P. Hartman (Baltimore, Md.)

5950:

Karimova, H. H. Geodesic flows in three-dimensional spaces of variable negative curvature. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 5, 3–14. (Russian)

In his famous monograph [Ber. Verh. Sächs. Akad. Wiss. Leipzig **91** (1939), 261–304; MR **1**, 243] E. Hopf examined geodesic flows in spaces of constant negative curvature, dividing such manifolds into two classes. In the first class, the set of directions at a fixed point for which the geodesics issuing from this point are wandering is of measure zero; the remaining manifolds are said to be of the second class. Hopf proved that the flow was ergodic (in fact, mixing) in manifolds of the first class and dissipative for the second class. He also extended these results (except for the mixing) to surfaces of variable negative curvature (and later, in Math. Ann. **117** (1940), 590–608 [MR **2**, 106], some positive curvature was admitted). The basic difficulty in the latter proofs was in establishing the measurability—trivial in the constant curvature case—of a geometrically described change of coordinates in the tangent bundle. The present author considers three-dimensional manifolds of variable negative curvature of a special kind; namely, there exists a point such that every geodesic surface through this point is totally geodesic. She sets up a change of coordinates in the tangent bundle completely analogous to Hopf's "asymptotic coordinates". She states without proof that the change of coordinates has a positive, continuous Jacobian. She repeats Hopf's arguments word for word to prove ergodicity or dissipativity in these manifolds of the first or second class respectively.

L. W. Green (Minneapolis, Minn.)

5951:

Moszyńska, M. A generalization of the Gauss-Bonnet formula. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 601–609. (Russian summary, unbound insert)

M. N. Stavroulakis generalized the Gauss-Bonnet formula to surfaces with "vertical points" as singularities [C. R. Acad. Sci. Paris **245** (1957), 1112–1114; MR **21** #1633]. In this paper the theorem is generalized to a wider class of surfaces. The author uses results of Stavroulakis and also results due to A. D. Alexandrow, *Die innere Geometrie der konvexen Flächen* [Akademie-Verlag, Berlin, 1955; MR **17**, 74].

T. J. Willmore (Liverpool)

5952:

Su, Buchin. On the theory of affine connections in an areal space. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2** (50) (1958), 185–190.

This is an expository lecture on the author's results already published partly in Sci. Record (N.S.) **1** (1957), 195–198 [MR **20** #7311], and partly in Sci. Sinica **6** (1957), 967–975 [MR **20** #7310]. *E. T. Davies* (Southampton)

GENERAL TOPOLOGY, POINT SET THEORY

See also 5536.

5953:

Vaidyanathaswamy, R. ★Set topology. 2nd ed. Chelsea Publishing Co., New York, 1960. vi+305 pp. $6.00.

A number of small improvements and corrections have been made to the 1st edition [Indian Math. Soc., Madras, 1947; MR **9**, 367], including a correction of the error mentioned at the end of the latter review. The preface refers to the paper of V. K. Balachandran, J. Madras Univ. Sect. B **28** (1958), 129–146 [MR **21** #4404] as supplementing the last chapter; and to the abstract of Hing Tong, Bull. Amer. Math. Soc. **54** (1948), 478–479, as solving a problem raised in the preface to the 1st edition.

5954:

Frolík, Zdeněk. An example concerning countably compact spaces. Czechoslovak Math. J. **10** (85) (1960), 255–257. (Russian summary)

The example shows that "regular" cannot be weakened to "Hausdorff" in the theorem that if every point-finite open covering of a regular space has a finite subcovering then the space is countably compact. *J. R. Isbell* (Seattle, Wash.)

5955:

Schweizer, B.; Sklar, A. Statistical metric spaces. Pacific J. Math. **10** (1960), 313–334.

Statistical metric (S.M.) spaces were introduced by Menger [Proc. Nat. Acad. Sci. U.S.A. **28** (1942), 535–537; MR **4**, 163]: the metric is defined in terms of distribution functions. Wald [see *Selected papers in statistics and probability by Abraham Wald*, McGraw-Hill, New York, 1955; MR **16**, 435] proposed an alternative generalized triangle inequality. This paper is concerned with various forms of the generalized triangle inequality and their consequences. It consists of three parts: (I) Axiomatics of S.M. spaces and comparison of the Menger and Wald forms; (II) Construction and study of some particular S.M. spaces—"equilateral", "simple", and "normal" S.M. spaces; (III) Neighborhoods, convergence, and continuity in S.M. spaces. *M. Loève* (Berkeley, Calif.)

5956:

Schweizer, B.; Sklar, A.; Thorp, E. The metrization of statistical metric spaces. Pacific J. Math. **10** (1960), 673–675.

Conditions under which a statistical metric space is a Hausdorff or a metric space; see #5955 above for background. *H. P. McKean, Jr.* (Cambridge, Mass.)

5957:

Schmidt, Jürgen. Eine Studie zum Begriff der Teilfolge. Jber. Deutsch. Math. Verein. **63**, Abt. 1, 28–50 (1960).

This paper describes a method of comparing functions such that all the standard generalizations of the concept "subsequence" are reformulated as specializations of this new notion. Begin with filter bases $\mathscr{A}$ and $\mathscr{B}$ of subsets of a set E and say that $\mathscr{A}$ refines $\mathscr{B}$, $\mathscr{A}\prec\mathscr{B}$, when each B in $\mathscr{B}$ contains at least one A in $\mathscr{A}$. Let f, and g, be relations between sets A_0, and B_0, and a fixed set E; let $\mathscr{A}$ and $\mathscr{B}$ be filters in A_0 and B_0 and define $f\mathscr{A}$ to be the filter basis in E of images under f of the sets of the filter $\mathscr{A}$. Then the pair $(f, \mathscr{A})$ is a "sequence". Call $(f, \mathscr{A})$ a "subsequence" of $(g, \mathscr{B})$ whenever $f\mathscr{A}$ refines $g\mathscr{B}$. Theorem 3: The following are equivalent statements about the pairs $(f, \mathscr{A})$ and $(g, \mathscr{B})$. (1) $(f, \mathscr{A})$ is a "subsequence" of $(g, \mathscr{B})$; that is, $f\mathscr{A}$ refines $g\mathscr{B}$. (2) $\mathscr{A}\prec f^{-1}g\mathscr{B}$. (3) There is a (perhaps multiple-valued) relation φ from (part or all of) A_0 into B_0 such that f contains the composite relation $g\varphi$ and $\mathscr{A}\prec\varphi^{-1}\mathscr{B}$.

After several other generalizations of the classical subsequence concept, the paper discusses Kelley's notion of "subnet of a net" [Duke Math. J. **17** (1950), 277–283; MR **12**, 194]. It shows that that definition can be made a specialization of this by translating Kelley's definition by means of the perfinality filter of the directed set of indices on which the net has been defined; that is, the filter generated by the sets $[a_0[=\{a|a\geq a_0\}$. Then Kelley's case becomes the result of assuming in (3) above that φ is single-valued and everywhere defined and that $f=g\varphi$.

The final sections extend the ideas first used by Tukey [*Convergence and uniformity in topology*, Princeton Univ. Press, Princeton, N.J., 1940; MR **2**, 67] of topologizing a directed system by using its perfinality filter as a neighborhood system for a point at infinity. *M. M. Day* (Urbana, Ill.)

5958:

Alfsen, E. M.; Fenstad, J. E. On the equivalence between proximity structures and totally bounded uniform structures. Math. Scand. **7** (1959), 353–360.

The title describes the content of the paper. Most of the results have been derived independently by other authors —as these authors point out. See I. S. Gál, Nederl. Akad. Wetensch. Proc. Ser. A **62** (1959), 304–326 [MR **21** #5944]. *H. H. Corson* (Seattle, Wash.)

5959:

Brown, Morton. Some applications of an approximation theorem for inverse limits. Proc. Amer. Math. Soc. **11** (1960), 478–483.

A map (continuous transformation) of a metric space to itself is called a 'near-homeomorphism' if it is arbitrarily close to a homeomorphism of the space onto itself.

Let X and Y denote compact metric spaces, and let f and g denote maps such that $f(X)\subset Y$, $g(Y)\subset X$, and both fg and gf are near-homeomorphisms. Then X is homeomorphic to Y. This follows quickly from the following theorem on inverse limit spaces: If, for each $i\geq 2$, f_i is a near-homeomorphism of the compact metric space X to itself, then the inverse limit space of the system (X, f_i) is homeomorphic to X. The proof rests on two quite interesting lemmas. The first says roughly that if (X_i, f_i) and (X_i, g_i) are inverse limit systems and f_i approximates g_i rather nicely (the measure of

the approximation being based upon the behavior of the sequence g_i), then the inverse limit space of (X_i, g_i) is the image of the inverse limit space of (X_i, f_i) under a rather natural map; and the second lemma places a restriction on the approximation which causes this map to be one-to-one.

Since a monotone map of a compact n-manifold $(n=1, 2)$ onto itself is a near-homeomorphism, the author's results include and extend a theorem due to C. E. Capel [Duke Math. J. **21** (1954), 233–245; MR **15**, 976]. Also if (in the first theorem above) f is onto and gf is a near-homeomorphism but fg is not, then one knows at least that $\dim X \leq \dim Y$.

F. B. Jones (Chapel Hill, N.C.)

5960:

Smirnov, Yu. M. Geometry of infinite uniform complexes and δ-dimension of point sets. Amer. Math. Soc. Transl. (2) **15** (1960), 95–113.

Translation of Mat. Sb. (N.S.) **40 (82)** (1956), 137–156 [MR **19**, 300].

ALGEBRAIC TOPOLOGY

See also 5536.

5961:

Trevisan, Giorgio. A proposito di un teorema di Petersen. Rend. Sem. Mat. Univ. Padova **30** (1960), 97–100.

Petersen's theorem states that all the edges of a trivalent graph G' can be colored with two colors, say A and B, so that the three edges at a vertex always consist of two A's and one B. Taking G' to be a planar graph, the author considers the dual graph G, which forms a map of triangles. He observes that the edges of G with the color A form a map of quadrangles, each having a B edge as a diagonal. He calls this reduction of G from a map of triangles to a map of quadrangles a 'Petersen decomposition' of G. According to P. G. Tait [see, e.g., W. W. R. Ball, *Mathematical recreations and essays*, 11th ed., Macmillan, London, 1947; MR **8**, 440], the famous four-color theorem is equivalent to the statement that the factorization of any planar trivalent graph G' can be carried out in such a way that all the polygons formed by the A edges have even numbers of sides. Then, of course, they can be colored alternately with two colors, say α and γ, so that the three edges of G' at each vertex will have three different colors: α, γ, and B. The author makes the obvious remark that this is equivalent to the possibility of making two 'Petersen decompositions' of G, such that all the diagonals of one are distinct from all the diagonals of the other.

H. S. M. Coxeter (Toronto)

5962:

Markov, A. A. Insolubility of the problem of homeomorphy. Proc. Internat. Congress Math. 1958, pp. 300–306. (Russian.) Cambridge Univ. Press, New York, 1960.

This paper differs from the author's note in Dokl. Akad. Nauk SSSR **121** (1958), 218–220 [MR **20** #4260] only in that a detailed description is given of the 4-manifold $M(S)$, assigned to each finite system S of generators and relations of a group. As the author points out, $M(S)$ is the manifold described in H. Seifert and W. Threlfall, *Lehrbuch der Topologie* [Teubner, Leipzig, 1934], Exercises 2 and 3, p. 180.

S. Mardešić (Zagreb)

5963:

Hilton, P. J.; Wylie, S. ★Homology theory: An introduction to algebraic topology. Cambridge University Press, New York, 1960. xv+484 pp. $14.50; 75s.

This admirable book is designed to make the transition between the elementary textbook and the research-level treatise in algebraic topology. It begins with concrete geometrical fundamentals and manages nevertheless to introduce its reader to enough of the heavyweight machinery of modern topology so that he may hope to attack the contemporary literature of the field.

The plan of the book is bipartite. The first portion is concerned principally with the homology theory of simplicial polyhedra, the second with singular homology. Each is preceded by a discursive list of background material, supposed to be familiar to the reader.

The "background" to Part I might reasonably be expected as part of the equipment of any beginning student of algebraic topology. It consists of the rudiments of analytic (i.e., set-theoretic) topology, the elements of abelian-group theory, and of Zorn's lemma. The material required for Part II is of a very different character. It includes the definitions and elementary properties of homotopy groups, function spaces and fibre spaces. References are given for the major theorems and outlines of proofs for the rest. However the authors' hope that "Propositions and corollaries should be deducible by the reader . . ." seems unduly optimistic.

The treatment of simplicial complexes differs from the traditional elementary discussion [cf. for example Pontryagin, *Foundations of combinatorial topology*, Graylock, Rochester, N.Y., 1952; MR **14**, 194] both in approach and in scope. On the one hand the theory of chain complexes is singled out for separate consideration and those results about simplicial complexes which are purely algebraic in character are accordingly proved in a purely algebraic manner. Moreover the homological algebra of abelian groups is introduced and it thus becomes possible to give a reasonable statement and proof of the algebraic part of the Künneth theorem (the geometric part being found in the second portion of the book).

The content of the usual elementary discussion is further enriched by the inclusion of the formal theory of cup and cap products. Also the notion of a block dissection of a simplicial complex is introduced, making practical the explicit computation of the homology groups and cohomology rings of several familiar spaces. Part I concludes with a chapter on the fundamental group and covering spaces. This contains only the elementary results, others being left to Part II.

Part II begins with an introduction to obstruction theory, confined to the elementary results except for a brief exposition of Steenrod's theory of maps of K^{n+2} into S^n which, however, does not attempt a definition of the Steenrod operations.

An exposition of both the simplicial and cubical singular homology theories follows. The equivalence is proved via acyclic models, which are then used to prove the

geometric part of the Künneth theorem, i.e., the Eilenberg-Zilber theorem. The Eilenberg-Steenrod axioms are demonstrated as theorems on singular homology.

Cup products are introduced formally and the Eilenberg-Zilber theorem is used to show the equivalence with Steenrod's geometric definition in terms of the diagonal map. As an application, the Hopf invariant is defined and its elementary properties deduced.

The final chapter is the most ambitious: it deals with spectral homology theory and with the cohomology of groups. The treatment of the former contains the derivation of the spectral sequence of a differential filtered graded group, but the theorem of Leray-Serre on the spectral sequence of a fibre space is stated without proof. There are, however, many applications. The transgression is defined and the Wang and Gysin sequences are derived, as well as the cohomology ring of ΩS^n. The Freudenthal suspension theorems are proved, and a few homotopy groups of spheres are computed.

Homology groups of groups are defined by means of projective resolutions of the integers. They are used to relate the equivariant homology of a covering space to the homology of the base space, thus recovering Hopf's theorem. Finally, the spectral sequence of a covering space is derived.

This brief account of the contents does not convey the richness of the discussion and the plenitude of examples which indeed are calculated not only to illuminate the systematic discussion but also to introduce the reader to the atmosphere of algebraic topology. This aim is even more evident in the very extensive collection of problems, which range from straightforward exercises to items out of the recent literature—at one point the reader is asked to define functional cup-products and prove their elementary properties!

At at least one point, this reviewer feels, the richness becomes excessive. The apparatus of elaborate notation and Zeeman diagrams seems more likely to convince the hapless student that spectral sequences are unintelligible than to start him on the perhaps unpleasant but certainly necessary job of learning for himself—from clean-cut definitions—what they are. This impression is fortified by the absence of a proof of the Leray-Serre theorem; such a proof would certainly be possible within the context of the book.

In any finite work on algebraic topology nowadays there must of course be omissions. It would for the most part be captious to criticize the author's choice of these. If Čech homology occurs only in an appendix this is no doubt justified by its closer connection with the theory of sheaves. CW-complexes are (surprisingly) not mentioned, but something of the same material is covered under the heading of infinite simplicial complexes. Most disappointing to the reviewer is the relegation of semi-simplicial complexes to a short set of problems. This surely slights their importance. But no doubt the reviewer is still in a minority in feeling that they constitute the only sensible approach to homotopy theory.

Similarly one might question the inclusion of such notions as abstract cell complexes as too highly specialized, and the parallel introduction of simplicial and cubical singular homology as redundant. But again these matters are certainly within the authors' discretion.

A particularly interesting feature of the book is its ambivalent attitude toward categories and functors, arrived at on what are no doubt reasonable pedagogical grounds. The authors are at some pains to show that their definitions are functorial. But these definitions are, apparently in the interest of concreteness, often made in a non-functorial way, and indeed the notions of category, functor and natural transformation appear first in the appendix to Chapter 9! As an example, the homology of a simplicial complex is defined before the formal introduction of chain complexes and thus the functorial factorization "simplicial complex→chain complex→homology" appears only a posteriori.

There are some idiosyncracies of terminology and notation which should be noted. The authors have adopted in part the suggestion of Postnikov and Boltyanskiĭ and refer to cohomology as "contrahomology" to accord with its variance. They have also introduced the picturesque notations $\Delta(X)$, $\square(X)$ for the simplicial and cubical singular chain complexes of X. Also the notations $A * B$ for $\mathrm{Tor}_1(A, B)$, $A \pitchfork B$ for $\mathrm{Hom}(A, B)$ and $A \dashv\!\vdash B$ for $\mathrm{Ext}^1(A, B)$ are used to emphasize the product-like character of these functors. These innovations are amiable enough, but the reviewer feels less sympathetic to the authors' decision to write maps sometimes to the right and sometimes to the left of their arguments. The scheme is to write maps on the right in the category of topological spaces and whenever they are determined therefrom by a covariant functor, and on the left when they come from contravariant functors. Reflection will show almost immediately that this is possible only at the expense of confusing associativity and commutativity in a category.

But these are minor quibbles. This book has in fact only one real flaw—a grossly inadequate bibliography listing only twenty-one books and entirely ignoring the periodical literature.

This is a badly needed book. It does an excellent job of carrying the serious beginning student of algebraic topology to a genuine acquaintance with the field, and seems to this reviewer likely to become a standard work in a domain where indeed it is essentially without a rival.

A. Heller (Urbana, Ill.)

5964:

Eckmann, B.; Hilton, P. J. Operators and cooperators in homotopy theory. Math. Ann. **141**, 1–21 (1960).

This is not a classification of homotopy-theorists but deals with the following topic. Let $\mathscr{C}$ be a category with "zero maps" in which finite sums and products (in the sense of Grothendieck) exist; the authors denote these by $A * B$ and $A \times B$ respectively and call them "free product" and "direct product". An object A together with a map $x: A \times A \to A$ is called a group, one with a map $x': A \to A * A$ is called a co-group, provided that x, x' are associative, have zero as unit and allow a right inverse; as is known, these ideas are easily expressed in categorical terms. Let $\mathscr{D}$ be the category whose objects are based topological spaces and whose maps are based homotopy classes of maps. In this category the loop space ΩX is a group and the suspension ΣX a co-group. Let $F \xrightarrow{j} X \xrightarrow{p} Y$ be a fibre map (in the sense of Serre) and $Y' \xrightarrow{i} X' \xrightarrow{q} F'$ be a co-fibre map (homotopy-extension condition true for i). Then, in the category $\mathscr{D}$, ΩY is a group of operators on F in a well-known manner and, dually, $\Sigma Y'$ a co-group of cooperators on F'; the notions "operator" and "co-

operator" again having been given the evident categorical definitions. These ideas permit the authors to use simple theorems on categories in the study of the homotopy theory of fibre spaces and co-fibre spaces; several applications are given. *V. Gugenheim* (Baltimore, Md.)

5965:

Whitehead, George W. Homology theories and duality. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 554–556.

A spectrum $\mathbf{E}$ is a sequence E_n of based spaces together with maps $f_n: E_n \to \Omega E_{n+1}$. Then the (reduced) cohomology groups of the based space X with respect to $\mathbf{E}$ are defined by

$$\bar{H}^n(X;\mathbf{E}) = \varinjlim [S^k X, E_{n+k}].$$

These groups satisfy the Eilenberg-Steenrod axioms, except for the dimension axiom, on the category of compact polyhedra. In this paper the author introduces (reduced) homology groups $\bar{H}_n(X;\mathbf{E})$ which satisfy the Eilenberg-Steenrod axioms, except for the dimension axiom, on compact polyhedra, and proves an Alexander duality theorem and a Poincaré duality theorem for π-manifolds (orientable manifolds whose fundamental class is stably spherical).

Let $X \circledast E_n$ be the 'smashed' product of X and E_n. The map f_n induces a map $g_n: X \circledast E_n \to \Omega(X \circledast E_{n+1})$ by the rule $g_n(x, y)(t) = (x, f_n(y)(t))$, $x \in X$, $y \in E_n$. Then we may define

$$\bar{H}_n(X;\mathbf{E}) = \varinjlim \pi_{n+k}(X \circledast E_k).$$

If $\mathrm{K}(\pi)$ is the Eilenberg-MacLane spectrum then, plainly, $\bar{H}_0(S^0;\mathrm{K}(\pi)) = \pi$, so that by the uniqueness theorem $\bar{H}_n(X;\mathrm{K}(\pi)) = \bar{H}_n(X;\pi)$ on compact polyhedra and we have a true generalization of classical homology theory.

The author introduces cup- and cap-products into the cohomology and homology theories described above and so proves the following results: (Alexander duality theorem) If A is a subcomplex of S^n, then $\bar{H}^p(A;\mathbf{E}) \cong \bar{H}_{n-p-1}(S^n - A;\mathbf{E})$. (Poincaré duality theorem) If X is a triangulable π-manifold of dimension n, then $H^p(X;\mathbf{E}) \cong H_{n-p}(X;\mathbf{E})$. *P. J. Hilton* (Birmingham)

5966:

Sitnikov, K. A. Combinatorial topology of nonclosed sets. I. The first duality law; spectral duality. Amer. Math. Soc. Transl. (2) **15** (1960), 245–295.

Translation of Mat. Sb. (N.S.) **34 (76)** (1954), 3–54 [MR **16**, 736].

5967:

Sitnikov, K. A. Combinatorial topology of nonclosed sets. II. Dimension. Amer. Math. Soc. Transl. (2) **15** (1960), 297–349.

Translation of Mat. Sb. (N.S.) **37 (79)** (1955), 385–434 [MR **17**, 1120].

5968:

Bokštein, M. F. Homology invariants of topological spaces. American Mathematical Society Translations, Ser. 2, Vol. 11, pp. 173–385. American Mathematical Society, Providence, R.I., 1959. iii + 385 pp. $5.90.

Translation of Trudy Moskov. Obšč. **5** (1956), 3–80; **6** (1957), 3–133 [MR **18**, 813; **19**, 875].

5969:

Postnikov, M. M. Investigations in the homotopy theory of continuous mappings. III. General theorems of extension and classification. American Mathematical Society Translations, Ser. 2, Vol. 11, pp. 115–153. American Mathematical Society, Providence, R.I., 1959. iii + 385 pp. $5.90.

Translation of Mat. Sb. (N.S.) **40 (82)** (1956), 415–452 [MR **18**, 753].

5970:

de Lyra, C. B. On spaces of the same homotopy type as polyhedra. Bol. Soc. Mat. São Paulo **12** (1957), 43–62 (1960).

By considering the realisations of the singular complex of a space in the manner of J. H. C. Whitehead and J. Milnor the author proves some theorems characterising spaces having the homotopy type of locally finite or finite polyhedra; perhaps the most interesting is the following. Theorem: Let x be a simply connected, pathwise connected space; then the following conditions are equivalent: (a) x is of the same homotopy type as a finite polyhedron; (b) $x \in \alpha$ and the groups $H_q(x)$ are finitely generated and vanish above a certain dimension; (c) $x \in \alpha$, the groups $\pi_q(x)$ are finitely generated for all $q \geq 0$, and Δx is finite; (d) x is dominated by a finite polyhedron. Here α denotes the class of pathwise connected spaces dominated by a C.W. complex and Δx the minimal dimension of such a dominating complex.

V. Gugenheim (Baltimore, Md.)

5971:

Miyazaki, Hiroshi. On realizations of some Whitehead products. Tôhoku Math. J. (2) **12** (1960), 1–30.

Let $\pi_1, \pi_2, \cdots, \pi_n, \cdots$ be a system consisting of a group π_1 and π_1-modules $\pi_2, \cdots, \pi_n, \cdots$. Then a classical theorem of J. H. C. Whitehead asserts that this system may be realized as the system of homotopy groups of a CW-complex. Theorem 1 of the present paper takes the realization problem a stage further by showing that if $q > p > 1$ and $T = T_{p,q}: \pi_p \otimes \pi_q \to \pi_{p+q-1}$ is an arbitrary homomorphism of π_1-modules, then the system may be realized by a CW-complex K in which T is the Whitehead product $[\pi_p(K), \pi_q(K)] \subseteq \pi_{p+q-1}(K)$ and all other Whitehead products of elements of dimension > 1 vanish.

In the course of establishing the apparatus for proving this theorem the author solves Problem 11 in Massey's list [Ann. of Math. (2) **62** (1955), 327–359; MR **17**, 653], in the sense that he shows that if X has a fixed point under the action of the group G then an n-connected fibre space X' over X with the desired properties exists. Theorem 1 may be generalized to give realization theorems for a set of Whitehead products $T_i = T_{p_i, q_i}$, provided the integers occurring in the different products of the set are sufficiently far apart in a sense made precise in the paper. The sense is such that the Jacobi identity is not involved.

The latter sections of the paper are concerned with the realizability of products $U_t: \pi_t \otimes \pi_t \to \pi_{2t-1}$. Theorem 3

establishes necessary and sufficient conditions for the realizability of U_2 under a very restrictive hypothesis on π_1 and π_2 which at least includes the case where π_1 is trivial. Theorem 4 gives necessary and sufficient conditions for the realizability of the system π_4, π_7, U_4; theorem 5, for the realizability of U_2 and $T_{3,3}$; theorem 6, for the realizability of the system π_1, π_6, π_{11}, U_6; and theorem 7, for that of π_1, π_7, π_{13}, U_7. Theorems 8, 9 and 10 give sufficient conditions for the systems π_3, π_5, U_3; π_5, π_9, U_5; and π_8, π_{15}, U_8 to be realizable.

The paper is not free from blemishes. Lemma 8 is false, so that theorem 4 is not proved in the generality claimed for it. Also relation (v) on p. 23 is false; it should read $[\alpha, [\alpha, \beta]] = [\alpha, \beta] \circ E\eta - [\alpha \circ \eta, \beta]$ [see Hilton, Proc. Cambridge Philos. Soc. **50** (1954), 189–197; MR **15**, 734; Theorem 6.1]. Thus theorem 5 is invalidated and the correct statement must be somewhat more complicated.

P. J. Hilton (Birmingham)

5972:

Iwata, Kôichi; Miyazaki, Hiroshi. Remarks on the realizability of Whitehead product. Tôhoku Math. J. (2) **12** (1960), 130–138.

A space of type $K(\pi, n; G, m; k; \cdots)$ is a space Y such that $\pi_n(Y) = \pi$, $\pi_m(Y) = G$, $m > n > 1$, $\pi_i(Y) = 0$, $i < m$, $i \neq n$, and $k \in H^{m+1}(\pi, n; G)$ is the Postnikov invariant.

Let Y be a countable CW-complex of type $K(\pi, n; G, 2n-1; k; \cdots)$. Then the Whitehead product U_n: $\pi \otimes \pi \to G$ (see preceding review) is given by

$$U_n = \Theta^*(\Psi^* - p_1^* - p_2^*)k,$$

where Ψ, p_1, p_2: $\pi \oplus \pi \to \pi$ are given by $\Psi(a, b) = a + b$, $p_1(a, b) = a$, $p_2(a, b) = b$, and $\Theta^*: H^{2n}(\pi \oplus \pi, n; G) \to \mathrm{Hom}(\pi \otimes \pi, G)$ is given by the Künneth formula [see A. H. Copeland, Jr., Proc. Amer. Math. Soc. **8** (1957), 184–191; MR **18**, 754]. Using this result, the authors are able to give simple necessary and sufficient conditions for the realizability of a given homomorphism $\pi \otimes \pi \to G$ as the Whitehead product U_n in a suitable space Y, for $2 \leq n \leq 5$.

P. J. Hilton (Birmingham)

5973:

Zisman, Michel. L'obstruction à la construction d'une section d'un fibré au sens de Kan. C. R. Acad. Sci. Paris **250** (1960), 646–647.

The author gives an analogue to the classical theory of obstructions to a cross-section applicable to Kan fibre-spaces; let $p: E \to B$ be such a fibre-space and let there be given a cross-section $g: B^1 \to E$ defined on the 1-section B^1 of B; then there is associated a local system of groups $\mathscr{F}_n(g)$ which associates with every $b \in B^0$ the group $\pi_n(F_b, g(b))$, where $F_b = p^{-1}b$.

Now, let $g: B^n \to E$ be a cross-section on the n-section; the author defines a cochain $c^{n+1}(g) \in C^{n+1}(B, \mathscr{F}_n(g))$ such that: $c^{n+1}(g) = 0$ if and only if g can be extended to B^{n+1}; $c^{n+1}(g)$ is a cocycle; $c^{n+1}(g)$ is natural for induced fibre-spaces.

V. Gugenheim (Baltimore, Md.)

5974:

Zisman, Michel. L'obstruction à la construction d'une section d'un fibré au sens de Kan. C. R. Acad. Sci. Paris **250** (1960), 793–794.

This note continues the theory of the one reviewed above. We adhere to the notations defined there.

For a technical purpose the author first proves a generalisation of the Eilenberg-Zilber theorem applicable to cohomology with local coefficients. Let g_0, g_1 be cross-sections of $p: E \to B$ defined on the n-section B^n and such that $g_0|B^{n-1}$, $g_1|B^{n-1}$ are homotopic by a homotopy k. The author defines the difference cochain

$$d(g_0, g_1; \bar{k}) \in C^n(B, \mathscr{F}_n(g_0)),$$

where $\bar{k}$ is an extension of k to $I \times B^n$ which "begins" at g_0. Then: (1) $\delta d(g_0, g_1; \bar{k}) = c^{n+1}(g_0) - \tau c^{n+1}(g_1)$, where $\tau: \mathscr{F}_n(g_1) \to \mathscr{F}_n(g_0)$ is induced by $k|I \times B^1$. (2) $d(g_0, g_1; \bar{k}) = 0$ if and only if $\bar{k}$ can be extended to a homotopy between g_0 and g_1. (There seems to be a misstatement at this point.) (3) $d(g_0, g_1; \bar{k})$ depends on k only; we denote it by $d(g_0, g_1; k)$. Thus all the essentials of the classical theory are recovered in this case; and the analogous theorems can be proved.

V. Gugenheim (Baltimore, Md.)

5975:

Shih, Weishu. Sur le théorème de Hurewicz-Fadell. C. R. Acad. Sci. Paris **250** (1960), 4095–4096.

The main objective of this note is the announcement of a generalization of a theorem of the reviewer and Hurewicz [Ann. of Math. (2) **68** (1958), 314–347; MR **21** #2237a]. However, the setting in this instance is the category of semisimplicial fiber bundles with structure group.

Let E denote a fiber bundle (semisimplicial) with base B, fiber F, and structure group G, whose associated principal bundle E_G is trivial over the n-skeleton of B (e.g., if B is n-connected). Furthermore, let $E_r^{p,q}$ denote the terms of the associated spectral sequence (homology). Then, employing the classifying space $\overline{W}(G)$, cohomology classes $\xi_r \in H^r(B, H_{r-1}(G))$, $n+1 \leq r \leq 2n$, are defined, which, together with the pairing $H_*(F) \otimes H_*(G) \to H_*(F)$ induced by the operation of G on F, give rise to cap product homomorphisms

$$\xi_r \frown : H_p(B, H_q(F)) \to H_{p-r}(B, H_{q+r-1}(F)).$$

Next, subgroups $D^r_{p,q}$, $R^r_{p,q}$ of $H_p(B, H_q(F))$ are defined, $n+2 \leq r \leq 2n$, in terms of the above cap product homomorphisms. The main result may be stated as follows. Theorem: The terms of the spectral sequence $E_r^{p,q}$, $r \leq 2n$, associated with the fiber bundle E, are given as follows: (i) $E_2^{p,q} = E_r^{p,q}$, $2 \leq r \leq n+1$; (ii) $d_{n+1} = \xi_{n+1} \frown$ (ξ_{n+1} the fundamental class of E); (iii) $E_r^{p,q} = D^r_{p,q}/R^r_{p,q}$, $n+2 \leq r \leq 2n$; (iv) $d_r(\alpha_0) = \xi_r \frown \alpha_0 + \xi_{r-1} \frown \alpha_1 + \cdots + \xi_{n+2} \frown \alpha_{r-n-2}$, $n+2 \leq r \leq 2n$, where $\alpha_0, \alpha_1, \cdots, \alpha_{r-n-2}$ is a sequence associated with $\alpha_0 \in D^r_{p,q}$.

E. R. Fadell (Madison, Wis.)

5976:

Browder, William. Homology and homotopy of H-spaces. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 543–545.

This note announces a remarkable study in the homology of H-spaces. In the theorems to be quoted, it is assumed that X is an arcwise-connected H-space such that $H_r(X)$ is finitely-generated for each r; and p denotes a prime number.

The author shows (theorem 2): If $H_m(X; Z_p) = 0$ for all sufficiently large m, then the first non-vanishing group $\pi_q(X) \otimes Z_p$ (with $q > 1$) occurs for q odd. In particular, $\pi_2(X) \otimes Z_p = 0$. This implies (theorem 3): If $H_m(X; Z) = 0$

for all sufficiently large m, then the first non-vanishing group $\pi_q(X)$ (with $q>1$) occurs for q odd. In particular, $\pi_2(X)=0$.

The last conclusion includes and generalises a result of E. Cartan on Lie groups; to do this had been a well-known problem for some time.

The author's methods depend on the "Bockstein spectral sequence" of X. This may be constructed as follows. Let Q be the group of rational numbers, and let Q_f be the group of fractions $m/p^f n$, where n is prime to p. If C is a free chain complex over the integers, we may form $C\otimes Q$ and filter it by the subcomplexes $C\otimes Q_f$. There results a spectral sequence with

$$E^1 = \sum_f H(C\otimes Q_f/Q_{f-1}) \cong \sum_f H(C\otimes Z_p).$$

The successive differentials d^r are the successive Bockstein boundaries. This spectral sequence, however, contains many redundant groups; if we identify $\{x\}\in E^r_f$ with $\{px\}\in E^r_{f-1}$ (for each possible x, r, f) we obtain a spectral sequence which is equivalent but simpler, and starts with $E^1=H(C\otimes Z_p)$. The author prefers to construct this by using an exact couple.

From an H-space X one can obtain "Bockstein spectral sequences" E^r in homology and E_r in cohomology, and these are dual and consist of Hopf algebras. The homology product need not be associative; if it is not, one defines powers by induction so that $h^n=h^{n-1}\cdot h$.

The line of proof is now as follows. Suppose given a primitive element $y\in E^r$ of dimension $2n+1$, such that $d^r y=x\neq 0$. Then the author constructs (theorem 1) a sequence of non-zero elements x_i in E^r and of dimension $2np^i$, so that $x_0=x$, and so that for each i one of the following alternatives holds: (1) $x_i=(x_j)^{p^j}$ with $j<i$; (2) there exist elements $\bar{x}_i$, $\bar{x}_j$ in E_r (with $j<i$) such that $\bar{x}_i x_i\neq 0$, $\bar{x}_j x_j\neq 0$ and $\bar{x}_i=(\bar{x}_j)^{p^j}$. This conclusion is evidently impossible if $H_m(X;Z_p)=0$ for all sufficiently large m.

The author's final theorem (theorem 4) states that if $H_m(X;Z)=0$ for all sufficiently large m, then X satisfies the Poincaré Duality Theorem.

J. F. Adams (Cambridge, England)

5977:

Swan, Richard G. Groups with periodic cohomology. Bull. Amer. Math. Soc. **65** (1959), 368–370.

The author sketches a proof that any finite group with periodic cohomology can act freely on a finite simplicial homotopy sphere. More precisely, he states (theorem 1): Let π be a finite group having periodic cohomology of period q. Let n be the order of π. Let d be the greatest common divisor of n and $\phi(n)$, ϕ being Euler's ϕ-function. Then π acts freely and simplicially on a finite simplicial homotopy $(dq-1)$-sphere X of dimension $dq-1$. Furthermore, if X is not required to be a finite complex, we can replace $dq-1$ by $q-1$.

The proof is in two parts: one has first to construct free resolutions of Z over π which are periodic, and then to realize these resolutions geometrically. The second part is easy, but the first part is hard. The author states (theorem 2): Let π, q, d be as in theorem 1. Then π has a projective resolution of period q and a free resolution of period dq. (One should perhaps add that the resolutions in theorem 2 are supposed to consist of finitely generated modules.)

In order to prove theorem 2, the author takes a trial resolution of Z over π, and writes A for the modules of cycles in dimension $q-1$. He has then to analyse the class of A, working at first modulo projective modules. The result on free resolutions involves an appeal to his previous paper [#5660 above, theorem 4]: this introduces the factor d. The paper closes with results (which arose in the course of the work) concerning the p-periods of a finite group.

J. F. Adams (Cambridge, England)

5978:

Swan, Richard G. A new method in fixed point theory. Comment. Math. Helv. **34** (1960), 1–16.

The author considers transformation groups (π, X), where π is a finite group and X a paracompact Hausdorff space every open covering of which has a finite-dimensional open refinement. It is assumed that each point of X is left fixed either by all the elements of the group or else by none except the identity. To study the properties of X^π, the set of fixed elements, the author uses a complete resolution W of π instead of an ordinary injective resolution. Assume first that X is a simplicial complex on which π acts simplicially. Let $J^n(X,G)=H^n(\mathrm{Hom}_\pi(W,C^*(X,G)))$. The basic theorem asserts that the inclusion $X^\pi\to X$ induces an isomorphism $J^*(X,G)\to J^*(X^\pi,G)$. The proof is simple. Passage to direct limits gives meaning to this result for X a space. (Actually, the author considers pairs (X,A) instead of just X, A being an invariant subset of X.) Applications of this result to the case $\pi=Z_p$, p a prime, give simple proofs for most of the known facts about X^π and its relation to X. A cup product, introduced into the first spectral sequence of the double complex

$$\mathrm{Hom}_\pi(W, C^*(X,G)),$$

yields information concerning the cohomology ring of X^π. For example, if $\pi=Z_p$, $G=Z_p$ and $X=S^{2m}\times S^{2n}$, and if π acts trivially on $H^*(X,Z)$ and $H^*(X,Z_p)$, then X^π, if connected, has the cohomology groups of $S^{2r}\times S^{2s}$ ($r\leqq m$, $s\leqq n$); and if $r\neq s$, its cohomology ring is that of $S^{2r}\times S^{2s}$.

P. A. Smith (New York)

DIFFERENTIAL TOPOLOGY

See also 5549, 5550, 5558.

5979:

★Colloque de Géométrie Différentielle Globale, tenu à Bruxelles du 19 au 22 décembre 1958. Centre Belge de Recherches Mathématiques. Librairie Universitaire, Louvain; Gauthier-Villars, Paris; 1959. 182 pp. FB. 300.

Papers by A. Lichnerowicz, R. Thom, P. Libermann, B. Segre, A. G. Walker, N. H. Kuiper, G. Reeb, G. Vranceanu, P. Dedecker, Ch. Ehresmann, A. van de Ven, J. Tits, G. Papy; to be reviewed individually.

5980:

Pontryagin, L. S. Smooth manifolds and their applications in homotopy theory. American Mathematical Society Translations, Ser. 2, Vol. 11, pp. 1–114. American Mathematical Society, Providence, R.I., 1959. iii+385 pp. $5.90.

Translation of Trudy Mat. Inst. Steklov. **45** (1955), 1–139 [MR **17**, 181].

5981:

Wu, Wen-Tsün. On Pontrjagin classes. III. American Mathematical Society Translations, Ser. 2, Vol. 11, pp. 155–172. American Mathematical Society, Providence, R.I., 1959. iii+385 pp. $5.90.

Translation of Acta Math. Sinica **4** (1954), 323–346 [MR **18**, 225].

5982:

Rodnyanskiĭ, A. M. Differentiable mappings and the order of connectivity. Amer. Math. Soc. Transl. (2) **15** (1960), 115–129.

Translation of Mat. Sb. (N.S.) **37 (79)** (1955), 69–82 [MR **17**, 288].

5983:

Clark, R. S.; Bruckheimer, M. R. Sur les structures presque tangentes. C. R. Acad. Sci. Paris **251** (1960), 627–629.

A differentiable manifold of dimension $2n$ is said to have an almost tangent structure if we are given a reduction of its tangent bundle to the group of matrices of the form

$$\begin{bmatrix} A & 0 \\ B & A \end{bmatrix} \quad (A \in \mathrm{GL}(n, R)).$$

An example of manifolds M with such structures is given by the tangent bundles of manifolds M'. The author gives a number of results concerning almost tangent structures.

M. F. Atiyah (Cambridge, England)

5984:

Hangan, T. Dérivées de Lie et holonomie dans la théorie des connexions infinitésimales. Acad. R. P. Romîne. Stud. Cerc. Mat. **11** (1960), 159–173. (Romanian. Russian and French summaries)

The author has announced some generalizations to arbitrary Lie groups of results of Lichnerowicz. [See A. Lichnerowicz, C. R. Acad. Sci. Paris **244** (1957), 1868–1870; MR **21** #4450; and T. Hangan, ibid. **247** (1958), 411–412; MR **21** #2273.] In this paper the generalizations announced are demonstrated. The work uses the theory of infinitesimal holonomy groups in the form explained by H. Ozeki [Nagoya Math. J. **10** (1956), 105–123; MR **18**, 232]. Also, at each point z of a space with a connection there is introduced a subgroup of the structure group whose Lie algebra is generated by the vertical components of the infinitesimal automorphisms of the group of automorphisms G. This subgroup is called the Kostant group corresponding to G at z.

A. Schwartz (New York)

5985:

Chern, Shiing-Shen. Complex analytic mappings of Riemann surfaces. I. Amer. J. Math. **82** (1960), 323–337.

The author takes the position that "complex function theory should be regarded as the first chapter of the theory of complex analytic mappings of complex manifolds". In this paper he implements this view by using methods of differential geometry to prove new and classical theorems about complex analytic mappings of a Riemann surface into a compact Riemann surface.

Let D be a compact differentiable oriented domain with boundary C and M a compact Riemann surface. Let $f\colon D \to M$ be a differentiable mapping and $n(a)$ the total order of f at $a \in M$, c the area of M and $v(D)$ the area of the image of D. Then it is proved that $n(a) + \int_{f(C)} \lambda = v(D)/c$, where λ is a certain differential form.

Now let f be a complex analytic mapping with stationary points of index m_i. Let K be the Gaussian curvature of M and Ω its element of area. Then

$$2\pi\chi(D) + \int_C \omega + 2\pi \sum (m_i - 1) = \iint_{f(D)} K\Omega,$$

where ω is a certain differential form. This generalizes a formula of A. Hurwitz.

Finally, when D is a compact Riemann surface with a finite number of points deleted, defect relations are obtained which generalize the classical relations of Nevanlinna-Ahlfors. *C. B. Allendoerfer* (Seattle, Wash.)

5986:

Nastold, Hans-Joachim. Über meromorphe Schnitte komplex-analytischer Vektorraumbündel und Anwendungen auf Riemannsche Klassen. I. Math. Z. **69** (1958), 366–394.

Es sei X eine Riemannsche Fläche und w ein meromorphes Differential auf X ($w \neq 0$). In dem Körper K der auf X meromorphen Funktionen erhält man mit Hilfe von w eine Differentiation ($f' = (df)/w$ für $f \in K$). Gegenstand der Untersuchungen sind zunächst Differentialgleichungen bezüglich dieser Differentiation:

$$(1) \quad y^{(q)} + k_1 y^{(q-1)} + \cdots + k_{q-1} y' + k_q y = 0 \quad (k_j \in K).$$

Der Punkt $a \in X$ heißt regulär, wenn nach Umschreibung von (1) bezüglich eines in a zentrierten uniformisierenden Parameters t die Koeffizienten der Differentialgleichung in a holomorph sind. Anderenfalls heißt a singulär. Der singuläre Punkt a heißt schwach singulär, wenn alle in gelochten Umgebungen von a definierten (im allgemeinen mehrdeutigen) Lösungen von (1) sich in a bestimmt verhalten. "f verhält sich bestimmt" bedeutet: Für reelle φ, ψ gibt es stets eine ganze Zahl m mit $f(t)t^m \to 0$ für $t \to 0$ in $\varphi < \arg t < \psi$. (Die Ungleichung ist in der universellen Überlagerung der in 0 gelochten t-Ebene zu lesen.) (1) heißt vom Fuchsschen Typ, wenn alle singulären Stellen schwach singulär sind. Das soll im folgenden immer vorausgesetzt werden. Die Menge S der singulären Stellen ist ohne Häufungspunkte in X und daher abzählbar. Die offene Menge $X - S$ der regulären Stellen soll mit $\bar{X}$ bezeichnet werden. Es sei $x_0 \in \bar{X}$. Die Keime der (holomorphen) Lösungen von (1) in der Umgebung von x_0 bilden einen Vektorraum C^q. Jede dieser Lösungen lässt sich in $\bar{X}$ beliebig holomorph fortsetzen. Man erhält so den Monodromie-Homomorphismus $\chi\colon \pi_1(\bar{X}, x_0) \to \mathrm{GL}(q, C)$ von (1): Verf. definiert die Gruppenoperation $\alpha \circ \beta$ in $\pi_1(\bar{X}, x_0)$ durch "erst Weg α, dann Weg β" und $\chi(\alpha)$ als die Matrix, die eine bestimmte Basis von C^q nach Durchlaufung von α in eine andere Basis transformiert. Die universelle Überlagerung H von $\bar{X}$ ist ein Hauptfaserbündel über $\bar{X}$ mit $\pi_1(\bar{X}, x_0)$ als Strukturgruppe. Vermöge

χ konstruiert Verf. ein zu H assoziiertes Vektorraum-Bündel $\overline{W}$ über $\overline{X}$ mit C^q als Faser. Verf. zeigt, daß man $\overline{W}$ auf natürliche Weise zu einem Vektorraum-Bündel W über X erweitern kann.

Es sei $\hat{K}$ der Ring der in H meromorphen Funktionen, die sich in den Punkten von S bestimmt verhalten. Zur Riemannschen Klasse R von (1) gehören alle Vektoren aus $\hat{K}^q$, die sich bei Decktransformationen von H gemäß der Monodromie χ verhalten. R ist eine Vektorraum über K. Verschiedene Fuchssche Differentialgleichungen (1) können dieselbe Riemannsche Klasse haben, die ja allein durch die Monodromie χ bestimmt ist. Genau diejenigen Vektoren aus R, deren Komponenten über C linear-unabhängig sind, sind Lösungsbasen von Fuchsschen Differentialgleichungen mit der Mondromie χ. Von einer Riemannschen Klasse kann man bereits reden, wenn nur eine Ausnahmemenge $S\subset X$ und eine Monodromie χ vorgegeben sind. Die Konstruktion von $\overline{W}$ und W geht für eine beliebige Riemannsche Klasse R. Es besteht ein Isomorphismus des K-Vektorraumes R und des K-Vektorraumes der meromorphen Schnitte von W. Verf. diskutiert ausführlich die Unterordnung der Riemannschen Klassen unter die allgemeineren holomorphen Vektorraum-Bündel. Sätze über Riemannsche Klassen ergeben sich aus den Sätzen über Vektorraum-Bündeln. [Vgl. H. Röhrl, Math. Ann. **133** (1957), 1–25; MR **19**, 274]. Die Konstruktion von W gelingt durch elementare lokale Betrachtungen, die an das bekannte Verhalten der Lösungsbasen von Differentialgleichungen in der Umgebung einer singulären Stelle anknüpfen.—Verf. will im zweiten Teil der Arbeit die Frage nach meromorphen Schnitten von W mit vorgegebenen Hauptteilen behandeln.
F. Hirzebruch (Bonn)

5987:

Nastold, Hans-Joachim. Über meromorphe Schnitte komplex-analytischer Vektorraumbündel und Anwendungen auf Riemannsche Klassen. II. Math. Z. **70** (1958), 55–92.

Im ersten Teil der Arbeit [#5986] wurden klassische Fragen der Funktionentheorie in Fragen nach meromorphen Schnitten holomorpher Vektorraum-Bündel übersetzt. Es sei W ein holomorphes Vektorraum-Bündel über der komplexen Mannigfaltigkeit X. Es sei $\mathfrak{M}^p(W)$ bzw. $\mathfrak{O}^p(W)$ die Garbe der lokalen meromorphen bzw. holomorphen p-Formen mit Koeffizienten in W. Ein Schnitt w der Quotientengarbe $\mathfrak{M}^p(W)/\mathfrak{O}^p(W)$ entspricht einem Cousindatum (Vorgabe von Hauptteilen meromorpher p-Formen). w kommt dann und nur dann von einer globalen meromorphen p-Form, wenn ein gewisses Hindernis, nämlich das Element $\delta_* w$ von $H^1(X, \mathfrak{O}^p(W))$, verschwindet. Die letzte Cohomologiegruppe verschwindet für Steinsche Mannigfaltigkeiten—insbesondere für nicht kompakte Riemannsche Flächen—, so dass dort diese additiven Cousinprobleme immer lösbar sind. Im kompakten Fall formuliert Verf. mit Hilfe des Serreschen Dualitätssatzes das Verschwinden der oben erwähnten Hindernisklasse in einer Weise, die für Anwendungen handlicher ist. Im Falle der Lösbarkeit wird eine Lösung des Cousinproblems explizit durch das Cousindatum angegeben: Unter Verwendung von Methoden von Kodaira (harmonic integrals) werden de Rhamsche Existenzsätze über Ströme (courants) mit vorgegebenen Singularitäten übertragen auf Ströme mit Koeffizienten in W. Die ganze Theorie wird vom Verf. schliesslich auf Riemannsche Flächen angewandt. Es folgen mehrere Existenzsätze, durch die klassische Fragen beantwortet werden.—Der Leser wird es begrüssen, dass Verf. die verwandten "modernen Methoden" in übersichtlicher Weise zusammenstellt.
F. Hirzebruch (Bonn)

5988:

Behnke, H.; Grauert, H. Analysis in non-compact complex spaces. Analytic functions, pp. 11–44. Princeton Univ. Press, Princeton, N.J., 1960.

This essay is a revised report of the one-hour lecture delivered by Prof. H. Behnke at the conference on analytic functions at Princeton, 1957. As a continuation of the report by the same author at the Congress in Amsterdam, 1954 [Proc. Internat. Congress Math. 1954, Amsterdam, vol. III, pp. 45–57, Noordhoff, Groningen, 1956; MR **19**, 170], the present one is a general survey on the recent developments in the field of complex analytic spaces.

The paper begins with a few examples, showing why generalizations of the notion of complex manifold are necessary in the study of analytic functions of several variables. The notion of complex analytic function is defined for several variables, almost the same as in the case of one variable. However, the complex analytic function of several variables has, in general, non-uniformizable points of ramification; e.g., $w=\sqrt{(z_1z_2)}$, or $w=\sqrt{(z_1z_2+z_0^2)}$. Hence, the generalization of the complex manifold is achieved by adjunction of certain singular points. Then come the notions of a finitely-sheeted analytically ramified covering space R and of holomorphic functions on R. Replacing the coordinate neighborhoods in the definition of a complex manifold by analytically ramified covering space, one has the notion of complex analytic structure. Notions such as holomorphic functions and analytic sets are defined similarly as on a manifold. Because of the existence of non-uniformizable singularities, there occur phenomena different from the case of non-singular manifolds. Some examples constructed by H. Grauert and R. Remmert are given; e.g., two pure $(n-1)$-dimensional analytic sets in an n-dimensional complex space can intersect in an isolated point.

The above generalization is due to Behnke, Stein, Grauert and Remmert. There is another method due to H. Cartan and J.-P. Serre [in Séminaire H. Cartan, 1953/54, Exp. 4–9; MR **19**, 577]. Recently Grauert and Remmert [Math. Ann. **136** (1958), 245–318; MR **21** #2063] have succeeded in proving that a complex space in the first sense is equivalent to a normal complex space in the latter sense.

Section 2 is devoted to results on the holomorphic mapping $\tau: X\to Y$, and the analytic decomposition $Z=Z(\tau)$ defined by τ:

$$X = \bigcup_x r_x(\tau), \quad r_x(\tau) = \tau^{-1}(\tau(x)), \quad x\in X.$$

Among the theorems given in this section (of course, without proof), the following one due to K. Stein [ibid. **132** (1956), 63–93; MR **18**, 649] seems to have particular importance: If $\tau: X\to Y$ is proper, i.e., the inverse image of a compact set of Y is compact, there is one and only one complex analytic structure in the complex quotient

space $X/Z(\tau)$ such that the natural projection $\varphi: X \to X/Z$ is holomorphic.

Section 3 is a general survey on modification. This notion comes from the compactification of the complex affine space C^n, on the one hand, and from the resolution of singularities, on the other hand. Let N be a thin set of co-dimension 1 in a complex space X, and $'N$ be a closed proper subset in another complex space $'X$. A continuous modification of X in N is a quintuple $('X, 'N, \tau, N, X)$ with the following properties. (1) τ is a biholomorphic mapping of $'X-'N$ onto $X-N$. (2) For any open covering $\{U_j\}$ of N, $\bigcup V_j$ is a neighborhood of $'N$, where V_j is the open kernel of $\tau^{-1}(U_j-U_j \cap N) \cup 'N$. A continuous modification is called genuine if N is non-empty, and if each $\tau^{-1}(x)$, $x \in N$, contains at least two points, provided that it is not empty. For every continuous modification which is not the identity, there exists exactly one equivalent genuine modification $('X, '\tilde{N}, \tau, \tilde{N}, X)$, where $'\tilde{N}$ is the set of degeneration of τ in $'X$. Further, there are given relevant expositions on topics such as proper modification, related (German, "verwandt") complex spaces, complex primitive space, and H. Hopf's σ-modification.

In Section 4, holomorphically complete spaces are discussed. Contrary to the case of complex 1-dimensional Riemann surfaces, there exist examples of compact complex manifolds which admit no meromorphic functions other than the constants, and complex manifolds homeomorphic to cells which admit no holomorphic functions other than the constants. To develop analysis on complex spaces, it is necessary to select a class of complex spaces which admit sufficiently many holomorphic functions on them. Such a class was introduced by K. Stein. A complex manifold M is said to be holomorphically complete or a Stein manifold, if it satisfies the following two conditions. (1) M is K-complete, i.e., for every point $x \in M$, there are a finite number of holomorphic functions $f_1, \cdots, f_k$ on M, such that in a neighborhood U of x, the set $f^{-1}(z) \cap U$, $z \in C^k$, always consists of isolated points. Here f means the mapping $M \to C^k$ defined by the set $(f_1, \cdots, f_k)$. (2) M is holomorphically convex, i.e., for every compact set B of X, the set of all points $x \in X$ for which all holomorphic functions f on X satisfy $|f(x)| \leq \sup |f(B)|$, is compact in X.

Some well-known theorems are given: for example, Cartan and Serre's fundamental theorem on coherent analytic sheaves over a Stein manifold, the Cousin problems, and the embedding of a Stein manifold into C^N. Further, the concept of holomorphic completeness is extended to complex spaces. Some of the results on Stein manifolds are generalized also to the latter case.

Section 5, titled "Plurisubharmonic functions", concerns topics on pseudoconvexity. The notion of plurisubharmonic functions is extended to a complex space due to the theorems on removable singularities by Grauert and Remmert [Math. Z. **65** (1956), 175–194; MR **18**, 475]. On every holomorphically convex complex space X, there exists a plurisubharmonic function $p(x)$ which tends to ∞ when x approaches the ideal boundary of X. Is the converse true? This is a generalization of the classical Levi's problem. It is noted that the second named author has succeeded in proving that this conjecture is true if $p(x)$ is strongly plurisubharmonic, while it is incorrect if $p(x)$ is assumed merely plurisubharmonic.

The final section 6 is devoted to the following theorem of Grauert on complex-analytic fibre bundles over a holomorphically complete space X [see Math. Ann. **133** (1957), 139–159, 450–472; **135** (1958), 263–273; MR **20** #4659, 4660, 4661]: Two analytic fibre bundles $\mathfrak{B}_1$ and $\mathfrak{B}_2$ with the same structure group are analytically equivalent if and only if they are topologically equivalent. Also, for every continuous fibre bundle $\mathfrak{B}$ over X with complex analytic fibre and group, there exists an analytic fibre bundle $\mathfrak{B}^*$ over X with the same fibre and group, which is equivalent to $\mathfrak{B}$.

Detailed references are given at the end of the paper.

S. Hitotumatu (Tokyo)

5989:

Holmann, Harald. Quotientenräume komplexer Mannigfaltigkeiten nach komplexen Lieschen Automorphismengruppen. Math. Ann. **139**, 383–402 (1960).

Let a Lie automorphism group L operate holomorphically on a complex analytic space X. Under what conditions does the quotient space X/L have a complex structure with the following natural properties? (Q_1) The canonical projection $\pi|X \to X/L$ is holomorphic. (Q_2) The ring of holomorphic functions over X/L is isomorphic to the ring of holomorphic L-invariant functions over X. For example, this is so, when the group L operates properly discontinuously on X. This fact is a well-known result due to H. Cartan [Séminaire H. Cartan 1953/54; see also his essay in *Algebraic Geometry and Topology*, pp. 90–102, Princeton Univ. Press, Princeton, N.J., 1957; MR **18**, 823]. The present paper is a systematic investigation on some generalizations of the above result of Cartan. Throughout the paper, the following condition is always assumed: (IE) ("Isotropie endlich") The isotropy group $G(x)=\{g: g \in L, g \circ x=x\}$ is finite for every point $x \in X$.

In general, the quotient space X/L with the quotient topology is not always a T_1-space, even when X is a T_2-space. Hence, the author first introduces the following conditions: (SLE) ("schwach lokal eigentlich") For every point $x \in X$, there exists a neighborhood U_x such that the mapping $\Phi|L \times U_x \to \mathscr{H}(U_x) = \Phi(L \times U_x)$, where $\Phi(g, x) \equiv g \circ x$ $(g \in L, x \in X)$, is proper (in the sense of Bourbaki, i.e., the inverse image of a compact set is compact). Further, if Φ is a proper map as $\Phi|L \times U_x \to X$, then the condition is called (LE) ("lokal eigentlich").

The author proves the following main results. Suppose that X is a complex manifold, and L is a Lie automorphism group on X with the properties (IE) and (SLE). Then the quotient space X/L is a T_1-space (not always a T_2-space), and one can introduce a pseudo-complex structure (similar to the usual complex structure except for the Hausdorff separation axiom) such that the above conditions (Q_1) and (Q_2) are fulfilled. Further, when L satisfies the condition (IE) and (LE), the quotient space X/L is a T_2-space and has a normal complex structure with the conditions (Q_1) and (Q_2). And finally, if the mapping Φ is proper as a mapping $L \times X \to X$, then the projection $\pi|X \to X/L$ is a proper map. The third fact is essentially due to a work by K. Stein [Math. Ann. **132** (1956), 63–93; MR **18**, 649].

The reviewer would like to mention two slight misprints indicated by the author himself. In line 13, p. 389, $(L \times X)\tau(G)$ should read $(L \times X)/\tau(G)$. In lines. 19–20, p. 397, "Eindeutigkeit" should read "Eineindeutigkeit".

S. Hitotumatu (Tokyo)

5990:

Hano, Jun-ichi; Kobayashi, Shoshichi. A fibering of a class of homogeneous complex manifolds. Trans. Amer. Math. Soc. **94** (1960), 233–243.

This paper is concerned with homogeneous complex manifolds G/B with an invariant volume element V. From V and the complex structure, a 2-form ρ of type (1, 1) is defined in the usual manner. Suppose that the elements of the Lie algebra $\mathfrak{g}$ of G are considered to be infinitesimal transformations over G/B. Let L denote the subgroup of G generated by the elements X of $\mathfrak{g}$ such that the interior product $\iota(X)\rho$ vanishes at the point o of G/B which corresponds to B. The following are proved: "(i) L is a closed subgroup containing B. (ii) L/B is a complex submanifold of G/B. (iii) G/L has an invariant symplectic structure. (iv) If L is compact, then L/B is a complex torus." Using these, the authors generalize the reviewer's results on homogeneous complex manifolds. It is shown that, for any compact Lie group G and any connected subgroup B with G/B even-dimensional, the coset space G/B has an invariant complex structure if and only if the semi-simple part of B coincides with that of the centralizer of a toral subgroup of G. A complete description of the invariant complex structure is also given.

H.-C. Wang (Evanston, Ill.)

5991:

Kodaira, K.; Spencer, D. C. On deformations of complex analytic structures. III. Stability theorems for complex structures. Ann. of Math. (2) **71** (1960), 43–76.

This paper continues the previous papers of the authors [same Ann. **67** (1958), 328–466; MR **22** #3009], and in particular gives the proofs of some of the results announced and used earlier.

In Part I of the present paper the authors give a systematic treatment of strongly elliptic systems of partial differential equations on a compact differentiable manifold which depend on several real parameters. Their results are then applied in Part II to prove the following basic results. (1) The principle of upper semi-continuity, i.e., $\dim H^q(V_t, \Omega(B_t))$ is an upper semi-continuous function of t, where V_t is a compact complex manifold varying differentiably with t, B_t is a holomorphic vector bundle on V_t (also varying differentiably with t), and Ω denotes the sheaf of germs of holomorphic sections. (2) If $\dim H^{r,s}(B_t)$ is independent of t ($H^{r,s}(B_t)$ denotes the space of harmonic forms on V_t of type (r, s) "with coefficients in B_t"), then the Green's operator and the harmonic operator depend differentiably on t. (3) $\sum(-1)^q \dim H^q(V_t, \Omega(B_t))$ is independent of t. (4) Any small deformation of a compact Kähler manifold is again a Kähler manifold. The proof of (4) uses rather different results in the theory of differential equations from those used in the proofs of (1)–(3).

M. F. Atiyah (Cambridge, England)

5992:

Kodaira, K. On deformations of some complex pseudo-group structures. Ann. of Math. (2) **71** (1960), 224–302.

In this paper the author extends the theory of deformations of complex structures developed in the joint papers with D. C. Spencer [cf. #5991] to the case of complex pseudo-group structures. Four classes of pseudo-groups are considered, their definitions being as follows. First the author defines the notion of a stable holomorphic or closed holomorphic differential form in the space of several complex variables. Roughly f is stable if any small variation of f is an apparent one caused by a small change of local complex coordinates. The pseudo-groups considered are then the pseudo-groups of all biholomorphic transformations in a domain which leave invariant (or invariant up to a non-vanishing factor) a stable holomorphic (or closed holomorphic) form.

If Γ is one of the preceding pseudo-groups then one can define the notion of a Γ-manifold, this being a complex manifold with coordinate transformations belonging to the given pseudo-group. If now V is a compact Γ-manifold the author introduces the sheaf Θ on V of germs of infinitesimal transformations belonging to Γ. In each of the four cases the author constructs a fine resolution of Θ and obtains analogues of the Dolbeault isomorphism. In particular this method shows that the cohomology groups $H^q(V, \Theta)$ are finite-dimensional. From here on the theory develops as in the complex case.

M. F. Atiyah (Cambridge, England)

5993:

Srinivasacharyulu, Kilambi. Sur certaines familles différentiables de G-structures. C. R. Acad. Sci. Paris **250** (1960), 1171–1173.

As far as the reviewer can make out, the purpose of this paper is to extend the theorem of Akizuki and Nakano [Proc. Japan Acad. **30** (1954), 266–272; MR **16**, 619] on the vanishing of certain cohomology groups of analytic sheaves from one complex manifold to a differentiable family of complex manifolds.

M. F. Atiyah (Cambridge, England)

5994:

Look, K. H. Slit space and extremal principle. Sci. Record (N.S.) **3** (1959), 289–294.

A compact metric space R is called a slit space if and only if (i) there is a non-empty closed subset S, called the slit, each point of which is a limit point of $R-S$, and (ii) $R-S=\varphi(X\times Y)$, where φ is a homeomorphism, X is a differentiable manifold of class C^2 called the base space and Y is a compact Hausdorff space called the side space. The author proves the following theorem: Suppose f is continuous and nonnegative on R, and suppose, for each $y\in Y$, $f[\varphi(x, y)]$ is of class C'' in x and satisfies $\Delta f[\varphi(x, y)] \geqq 0$ on X, Δ being the Beltrami operator with respect to some Riemannian metric on X; then f takes on its maximum on the slit S.

A slit space R is said to be partly analytic if and only if $R\subset V_n$, a complex analytic Hermitean manifold, its base space W is a complex analytic manifold of dimension $m\geqq 1$, its side space T is a real compact differentiable manifold, and φ is a diffeomorphism which is analytic on W. The author concludes with the following theorem: Suppose D_1 is a domain in V_n whose closure $\bar{D}_1$ is compact, let $\bar{D}_2=\partial D_1$, and suppose $\bar{D}_1, \cdots, \bar{D}_{k-1}$ are partly analytic slit spaces in which $\bar{D}_{i+1}$ is the slit of $\bar{D}_i$. Then if f is continuous on $\bar{D}_1$ and analytic on D_1, f takes on its maximum on $\bar{D}_k$.

C. B. Morrey, Jr. (Berkeley, Calif.)

[illegible]

Kodaira, K.; Spencer, D. C. On deformations of complex analytic structures. III. Stability theorems for complex structures. Ann. of Math. (2) 71 (1960), 43–76.

[illegible]

[illegible] A. Weil (Princeton, N. J.)

[illegible]

AUTHOR INDEX

PART A

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